MAE2103 - Engineering Mechanics I
Course Notes
Prof. Brandon Runnels

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0 Introduction

Welcome to Engineering Mechanics I. This class is usually referred to as “Statics,” but we’ll be covering some extra material that typically falls into the category of “Dynamics.” For the majority of this class, we will be looking at mechanical systems that do not move, or are in “static equilibrium.”

0.1 Overview

The majority of the course (15 weeks) will be spent on the Statics portion of the class. The governing equations of statics are:

$$\sum F = 0 \quad \sum M = 0$$

(0.1)

where \( F \) are the force vectors and \( M \) are the moment vectors. In other words, “the sum of the forces and moments are equal to zero.”

For dynamics, the governing equations are similar, except that we have time dependence. The governing equations become:

$$\sum F = \frac{dP}{dt} \quad \sum M = \frac{dL}{dt}$$

(0.2)

where \( P \) is the momentum vector and \( L \) is the angular momentum vectors.

0.2 Units

Let’s review some of the basic units that we will use in this course:

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<th>Metric</th>
<th>USCS</th>
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<tr>
<td>Length</td>
<td>( m )</td>
<td>ft</td>
</tr>
<tr>
<td>Time</td>
<td>( s )</td>
<td>s</td>
</tr>
<tr>
<td>Mass</td>
<td>( kg )</td>
<td>slug</td>
</tr>
<tr>
<td>Temperature</td>
<td>( ^\circ C, K )</td>
<td>( ^\circ F, ^\circ R )</td>
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<td>Area,volume</td>
<td>( m^2, m^3 )</td>
<td>ft(^2, ) ft(^3 )</td>
</tr>
<tr>
<td>Velocity, acceleration</td>
<td>( m/s )</td>
<td>ft/s, ft/s(^2 )</td>
</tr>
<tr>
<td>Force</td>
<td>( N = \frac{kgm}{s^2} )</td>
<td>( lb = \frac{slug ft}{s^2} )</td>
</tr>
<tr>
<td>Pressure</td>
<td>( Pa = \frac{N}{m^2} )</td>
<td>( psi = \frac{lb}{in^2} )</td>
</tr>
<tr>
<td>Energy</td>
<td>( J = Nm )</td>
<td>ft lb(^\prime )</td>
</tr>
<tr>
<td>Power</td>
<td>( W = \frac{j}{s} )</td>
<td>ft lb/s</td>
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Let’s make some notes about units. Specifically, note that units:

- can be multiplied, divided, cancelled, even square-rooted
- cannot be added or subtracted

Why do we care about units? Not only do they connect numbers to physically meaningful quantities, but they are a great way to check your answer. If you find that you are adding different units together, or that the units of your answer are wrong, you’ve probably made a mistake somewhere.
1 Linear Algebra

Linear algebra is a deep and elegant branch of mathematics that has a wide variety of applications. In mechanics, linear algebra is a framework for describing quantities in a systematic way that makes analysis easy. In this section: introduce only the linear algebra that will be useful in this course.

1.1 Vectors

Before defining vectors, we need to define scalar quantities:

**Definition 1.1.** A **scalar** quantity is specified by its magnitude only

What are some examples of scalar quantities?

- mass \((m)\)
- time \((t)\)
- volume \((V)\)
- speed \((s)\)

What is a vector?

**Definition 1.2.** A **vector** quantity is specified by both a magnitude and a direction. A vector quantity is expressed as \(\vec{v}, \vec{v}, \vec{v}, \theta\)

What are some examples of vector quantities?

- position \((x)\)
- velocity \((v)\)
- force \((f)\)

Notice how I drew the angle with respect to a dotted line. Why draw the dotted line like that, instead of straight down, or up? I implicitly chose a **coordinate system**.

1.1.1 Coordinate system

**Definition 1.3.** A **coordinate system** is a convention for measuring locations in space.

Examples:
Notes:

- There are 3D polar coordinate systems too (spherical and cylindrical) but they are ugly and horrible and not too useful right now. In general, we can get along pretty well by simply using trigonometry.
- The number of dimensions must always match the number of coordinate variables.
- When working in 3D, we use a right-handed system.

It’s really important to get comfortable with this because it will be used a lot when we get to moments.

1.1.2 Component representation

There are two ways of representing vectors in component representation:

- matrix form
- unit vector form

Example 1.1
Write the following figure in component notation

First: we need to define a coordinate system. Let the upward pointing axis be \( z \). Then the vector is

\[
x = \begin{bmatrix} 2m \\ 2m \\ 4m \end{bmatrix}
\]  

(1.1)

### 1.1.3 Converting to/from angular representation

Because of the way problems are specified, it is often necessary to convert to and from magnitude-angle notation to coordinate notation (and back)

**Example 1.2**

Convert the vector to rectangular coordinates:

The \( x \) component is \( L \cos \theta \) and the \( y \) component is \( L \sin \theta \) so the resultant vector is

\[
x = \begin{bmatrix} 4m \\ 4\sqrt{3}m \end{bmatrix}
\]  

(1.2)

We will have to do this for 3D vectors as well:
Example 1.3

Find the component representation of the following vector:

We can find the z component directly:

\[ z = (1\,\text{m}) \sin(60^\circ) = \frac{\sqrt{3}}{2}\,\text{m} \]  

(1.3)

It is more tricky to find the x and y magnitudes. First we have to find the magnitude of the "projected" vector, which is \((1\,\text{m}) \cos(60^\circ) = 1/2\,\text{m}\). Now, we can compute the x and y magnitudes:

\[ x = 1/2\,\text{m}\cos(45^\circ) = \frac{1}{2\sqrt{2}}\,\text{m} \quad y = 1/2\,\text{m}\cos(45^\circ) = \frac{1}{2\sqrt{2}}\,\text{m} \]  

(1.4)

So the component representation is

\[ \mathbf{x} = \begin{bmatrix} 1/2\sqrt{2}\,\text{m} \\ 1/2\sqrt{2}\,\text{m} \\ \sqrt{3}/2\,\text{m} \end{bmatrix} \]  

(1.5)

1.1.4 Vector algebra

The nice thing about vectors (especially in the component notation) is that we can add, subtract, and scale them easily.

- Vector addition:

\[ \mathbf{a} = \begin{bmatrix} a_x \\ a_y \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} b_x \\ b_y \end{bmatrix} \]  

(1.6)

then I can add them together as

\[ \mathbf{c} = \mathbf{a} + \mathbf{b} = \begin{bmatrix} a_x + b_x \\ a_y + b_y \end{bmatrix} \]  

(1.7)

Pictorally:
Vector subtraction:

\[ \mathbf{c} = \mathbf{a} - \mathbf{b} = \left[ \begin{array}{c} a_x - b_x \\ a_y - b_y \end{array} \right] \]  

Note that in the picture, it is clear that \( \mathbf{a} = \mathbf{b} + \mathbf{c} \).

Scalar multiplication:

\[ \alpha \mathbf{a} = \left[ \begin{array}{c} \alpha a_x \\ \alpha a_y \end{array} \right] \]

Pictorially, multiplying by a scalar simply changes the magnitude of the vector by the value of that scalar.

What about vector multiplication?

1.1.5 Unit vectors

Since we can add, subtract, and multiply vectors, we can define the following:

\[ \hat{\mathbf{i}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \hat{\mathbf{j}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \hat{\mathbf{k}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \]

Then we can write the vector in the above example as

\[ \mathbf{x} = (2m)\hat{\mathbf{i}} + (2m)\hat{\mathbf{j}} + (2m)\hat{\mathbf{k}} \]

I prefer to avoid using unit vectors (so I probably won’t use them much in lecture) but there’s nothing wrong with them and you should feel free to use them if you like.
1.2 Matrices

Matrices (and vectors) are useful tools for organizing the equations that we have to solve.

**Definition 1.4.** A matrix is a set of numbers organized in a grid

(It’s a simple definition but it will do for now.)

Examples of matrices:

\[
\begin{bmatrix}
1 & 2 \\
3 & 4 \\
\end{bmatrix}
\quad \text{2x2}
\]

\[
\begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9 \\
\end{bmatrix}
\quad \text{3x3}
\]

\[
\begin{bmatrix}
9 & 3 & 2 & 1 \\
0 & 3 & 1 & 2 \\
\end{bmatrix}
\quad \text{2x4}
\]

Notes:

- \([\# \text{ rows}] \times [\# \text{ columns}]

- \(n\times1\): n-D vectors

- \(n\times n\): square

- Notation: \(A\)

- Addition/subtraction/scalar multiplication works if same size

### 1.2.1 Matrix matrix multiplication

Matrices \(A\) and \(B\) can be multiplied if \(\# \text{ columns of } A = \# \text{ rows of } B\)

\[
\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1q} \\
a_{21} & a_{22} & \cdots & a_{2q} \\
\vdots & \vdots & \ddots & \vdots \\
a_{p1} & a_{p2} & \cdots & a_{pq} \\
\end{bmatrix}
\begin{bmatrix}
b_{11} & b_{12} & \cdots & b_{1r} \\
b_{21} & b_{22} & \cdots & b_{2r} \\
\vdots & \vdots & \ddots & \vdots \\
b_{q1} & b_{q2} & \cdots & b_{qr} \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
\sum_{i} a_{1i}b_{i1} & \sum_{i} a_{1i}b_{i2} & \cdots & \sum_{i} a_{1i}b_{ir} \\
\sum_{i} a_{2i}b_{i1} & \sum_{i} a_{2i}b_{i2} & \cdots & \sum_{i} a_{2i}b_{ir} \\
\vdots & \vdots & \ddots & \vdots \\
\sum_{i} a_{pi}b_{i1} & \sum_{i} a_{pi}b_{i2} & \cdots & \sum_{i} a_{pi}b_{ir} \\
\end{bmatrix}
\]

(1.12)

The result is:

\[
\begin{bmatrix}
(1)(2) + (3)(1) + (2)(1) & \cdots & (1)(0) + (3)(3) + (2)(4) \\
(8)(0) + (1)(2) + (3)(5) & \cdots & (8)(0) + (1)(2) + (3)(5) \\
\end{bmatrix}
\quad \text{2x2}
\]

(1.13)

\[
\begin{bmatrix}
7 & 14 \\
17 & 51 \\
\end{bmatrix}
\]

(1.14)

Example 1.4

\[
\begin{bmatrix}
1 & 3 & 2 \\
0 & 5 & 9 \\
2 & 1 & 3 \\
\end{bmatrix}
\quad \begin{bmatrix}
2 & 0 \\
1 & 3 \\
1 & 4 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
(1)(2) + (3)(1) + (2)(1) & (1)(0) + (3)(3) + (2)(4) \\
(8)(0) + (1)(2) + (3)(5) & (8)(0) + (1)(2) + (3)(5) \\
\end{bmatrix}
\quad \begin{bmatrix}
7 & 14 \\
17 & 51 \\
\end{bmatrix}
\]

(1.15)

Example 1.5

\[
\begin{bmatrix}
3 & 1 & 4 \\
8 & 1 & 3 \\
1 & 4 & 0 \\
\end{bmatrix}
\quad \begin{bmatrix}
0 & 2 \\
5 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
(3)(0) + (1)(2) + (4)(5) & (8)(0) + (1)(2) + (3)(5) \\
(1)(0) + (4)(2) + (0)(5) & \end{bmatrix}
\quad \begin{bmatrix}
22 & \\
17 & \\
\end{bmatrix}
\]

(1.16)
Notice that the matrix “ate” the vector and turned it into another vector. This is actually a very nice way of looking at matrices: we can think of them as machines that turn a vector into another vector.
1.2.2 Determinants

You may have (hopefully) seen the determinant of a matrix before. It may seem like a rather odd quantity, but its usefulness will become apparent as we use it in the course. In particular, we will use it a lot when computing cross products and when solving linear systems. A rigorous definition of the determinant is beyond the scope of this course, so instead we will introduce it by example.

**Example 1.6**

For a 2x2 matrix:

\[
\begin{vmatrix}
1 & 2 \\
3 & 4
\end{vmatrix} = (1)(4) - (2)(3) = 4 - 6 = -2
\]  

(1.17)

**Example 1.7**

For a 3x3 matrix: two ways of computing

\[
\begin{vmatrix}
1 & 2 & 3 \\
4 & 5 & 2 \\
0 & 3 & 1
\end{vmatrix} = (1)((5)(1) - (2)(3)) - (2)((4)(1) - (2)(0)) + (3)((4)(3) - (5)(0))
\]

\[= (1)(-1) - (2)(4) + (3)(12) = 27 \]  

(1.18)

Alternatively, we can compute the diagonals

\[
(1)(5)(1) + (2)(2)(0) + (3)(4)(3) - (1)(2)(3) - (3)(5)(0) - (2)(4)(1) = 27
\]  

(1.19)

Let’s make a few notes about the properties of the determinant. Again, these properties will come in handy later on.

- If the vectors that make up two of the columns of the matrix are parallel, then the determinant will be zero.
- If the third vector is equal to a combination of the first two, the determinant will be zero.
- If \(A\) is a \(n \times n\) matrix and \(\alpha\) is a scalar, then \(\det(\alpha A) = \alpha^n \det(A)\)

### 1.3 Dot product

Suppose we have two vectors:

\[
a = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \quad b = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}
\]  

(1.21)

How can we multiply them together?

\[
\begin{bmatrix} 1 \\ 2 \\ 3 \\
4 \\ 5 \\ 6
\end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\
4 & 5 & 6
\end{bmatrix} \quad \quad \text{bad} \quad \quad \text{good!}
\]  

(1.22)
This is called the dot (or inner) product, and it gives a scalar value:

\[
\begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 6
\end{bmatrix} = (1)(4) + (2)(5) + (3)(6) = 32
\]  

This operation is called the \textbf{dot product} and is denoted by

\[a \cdot b\]

\subsection{Angular representation}

In polar form the dot product is computed simply as

\[a \cdot b = (\text{magnitude of } a)(\text{magnitude of } b) \cos(\text{angle between } a \text{ and } b)\]

\subsection{Vector norm}

We know how to convert from angular to component representation; how do we convert from component to polar?

If we have a vector

\[\mathbf{v} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}\]

then the magnitude is

\[\sqrt{v_x^2 + v_y^2 + v_z^2} = \sqrt{\mathbf{v} \cdot \mathbf{v}} \equiv ||\mathbf{v}||\]

This is called the \textbf{vector norm}. Some properties of the norm are:

\begin{itemize}
  \item \(||\mathbf{v}|| \geq 0\)
  \item \(||\mathbf{v}|| = 0\) if and only if \(\mathbf{v} = \mathbf{0}\)
  \item \(||\alpha \mathbf{v}|| = |\alpha||\mathbf{v}||\)
  \item \(||\mathbf{u} + \mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||\) (the triangle inequality)
\end{itemize}

With this terminology, we can write this very useful equation:

\[\mathbf{u} \cdot \mathbf{v} = ||\mathbf{u}|| ||\mathbf{v}|| \cos \theta\]
1.3.3 Unit vectors

Whenever we have a nonzero vector, we can divide by its magnitude to get a unit vector. Given \( \mathbf{v} \) as we had earlier, then we define

\[
\hat{\mathbf{v}} = \frac{1}{|| \mathbf{v} ||} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}
\]  

(1.29)

**Example 1.8**

Compute the unit vector corresponding to the position vector

\[ \mathbf{x} = \begin{bmatrix} 1 \text{m} \\ 2 \text{m} \\ 3 \text{m} \end{bmatrix} \]  

(1.30)

The magnitude is

\[ || \mathbf{x} || = \sqrt{1^2 + 4^2 + 9^2} = \sqrt{14^2} = 14 \text{m} \]  

(1.31)

(Note that the units are in meters.)

Now we divide:

\[ \hat{\mathbf{x}} = \begin{bmatrix} 1/14 \\ 2/14 \\ 3/14 \end{bmatrix} \]  

(1.32)

Note that the unit vector is (ironically) unitless. This should always be true of unit vectors.

Unit vectors are very nice because they allow us to decouple the magnitude of a vector from its direction.

**Example 1.9**

Consider a cable that is attached to a point on a wall as shown.

The cable is pulled so that it has a tension of \( t \). Compute the force vector acting by the cable on the wall.

We will do this sort of thing a lot in this class! The thing to in this situation is recognize that we can write the force vector in this way

\[
\mathbf{f} = t \mathbf{n}
\]  

(1.33)

where \( t = || \mathbf{f} || \), the magnitude of the force. We already know \( t \) – it was given to us. Now, we need to compute \( \mathbf{n} \). How do we do this? We know that the force will be in the same direction of the cable. So, we can find the unit vector that is parallel to the cable, and it will serve as a unit vector for the force too. The vector going along
the cable is
\[
d = \begin{bmatrix} 3m \\ 4m \\ 0 \\ 5m \end{bmatrix} - \begin{bmatrix} 4m \\ 0 \\ 3m \\ -5m \end{bmatrix} = \begin{bmatrix} -1m \\ 4m \\ 0 \\ -8m \end{bmatrix}
\]
\[\text{(1.34)}\]

(Remember to be careful about the signs!) Now, we just need to normalize \(d\):
\[
||d|| = \sqrt{(3m)^2 + (4m)^2 + (5m)^2} = \sqrt{25m^2 + 25m^2} = 5\sqrt{2}m
\]
\[\text{(1.35)}\]

so our unit vector is
\[
n = \frac{d}{||d||} = \begin{bmatrix} 4/5\sqrt{2} \\ 3/5\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}
\]
\[\text{(1.36)}\]

and our force vector is
\[
f = t n = \begin{bmatrix} 4t/5\sqrt{2} \\ 3t/5\sqrt{2} \\ -t/\sqrt{2} \end{bmatrix}
\]
\[\text{(1.37)}\]

1.3.4 Projection

The dot product allows us to compute the projection of one vector onto another. It determines the “amount” of the vector that is in the direction of the other. For example:

The projection is \(||u|| \cos \theta\). We can write this in terms of the dot product:
\[
||u|| \cos \theta = \frac{||u||||v|| \cos \theta}{||v||} = \frac{u \cdot v}{||v||} = u \cdot \hat{v}
\]
\[\text{(1.38)}\]

Example 1.10

Consider a train moving along a track, subjected to a constant force:
What is the effective force that is acting in the direction of the motion of the car?

\[ \Delta x = x_f - x_i = \begin{bmatrix} 1m \\ 1m \end{bmatrix} \]  

The projection is:

\[
\frac{\Delta x \cdot f}{||\Delta x||} = \frac{(1m)(0N) + (1m)(1N)}{\sqrt{2}m} = \frac{1}{\sqrt{2}}N
\]  

Note that the unit is Newtons, which is what we expect.

## 1.4 Cross product

We will also make extensive use of the vector cross product, especially when we start talking about moments. Here, we will just define it and do an example.

### 1.4.1 Component representation

Suppose we have two vectors

\[
\mathbf{u} = \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}
\]

The cross product between them is

\[
\mathbf{u} \times \mathbf{v} \equiv \begin{bmatrix} u_yv_z - u_zv_y \\
 u_zv_x - u_xv_z \\
 u_xv_y - u_yv_x \end{bmatrix}
\]

Fortunately, there's a nice mnemonic for remembering how to compute the cross product using the \( \hat{i}, \hat{j}, \hat{k} \) unit vectors:

\[
\mathbf{u} \times \mathbf{v} = \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{bmatrix} = \hat{i}(u_yv_z - u_zv_y) + \hat{j}(u_zv_x - u_xv_z) + \hat{k}(u_xv_y - u_yv_x)
\]

Pictorially:
Notes:

- $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$ (you’ll show this in your homework)
- How do we know which way it points? Right hand rule.
- And $\mathbf{v} \times \mathbf{u} = -\mathbf{u} \times \mathbf{v}$

**Example 1.11**

Compute the cross product between

$$ \mathbf{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} $$

What do you think the cross product will be? It should be zero because $\mathbf{b} = 2\mathbf{a}$ which means they point in the same direction.

Let’s find out:

$$ \mathbf{a} \times \mathbf{b} = \begin{bmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{bmatrix} = \mathbf{i} (12 - 12) + \mathbf{j} (6 - 6) + \mathbf{k} (4 - 4) = 0 $$

exactly as we expected!

### 1.4.2 Scalar cross product

Unlike the dot product, the cross product does not generalize naturally to 2D (or any other dimension, for that matter). When working with 2D vectors, we can write them in 3D:

$$ \mathbf{u} = \begin{bmatrix} u_x \\ u_y \\ 0 \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} v_x \\ v_y \\ 0 \end{bmatrix} $$

Then their cross product is

$$ \mathbf{u} \times \mathbf{v} = \begin{bmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{bmatrix} = \mathbf{k} (u_x v_y - u_y v_x) $$

2D cross products have a $\mathbf{k}$ component only, so just keep track of the magnitude:

$$ \|\mathbf{u} \times \mathbf{v}\| = \begin{bmatrix} u_x \\ u_y \\ v_x \\ v_y \end{bmatrix} $$
1.4.3 Angular representation

This is where component representation really shines, because there’s just no easy way to compute the cross product in angular representation. However, we do have the following important identity:

\[ ||u \times v|| = ||u|| ||v|| \sin \theta \]  

(1.50)

Notes:

- When is the magnitude of the cross product maximized? When \( u \) and \( v \) are orthogonal (perpendicular) to each other.
- When is the magnitude of the cross product minimized? When \( u \) and \( v \) are parallel to each other. (compare to dot product!)

Example 1.12

Given

\[
\begin{bmatrix}
6 & 6 & 1 \\
1 & 5 & 3
\end{bmatrix}
\]

So

\[
\begin{bmatrix}
\hat{i} & \hat{j} & \hat{k} \\
6 & 6 & 1 \\
1 & 5 & 3
\end{bmatrix} = \begin{bmatrix}
18 - 5 \\
-18 \\
30 - 6
\end{bmatrix} = \begin{bmatrix}
13 \\
-17 \\
24
\end{bmatrix}
\]

(1.52)

Example 1.13

Compute the cross product between \( \hat{i} \) and \( \hat{j} \).

What do you think it will be?

\[
\hat{i} = \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}, \quad \hat{j} = \begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix}
\]

(1.53)

\[
\hat{i} \times \hat{j} = \begin{bmatrix}
\hat{i} & \hat{j} & \hat{k} \\
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix} = \hat{i}(0) + \hat{j}(0) + \hat{k}(1) = \hat{k}
\]

(1.54)

1.4.4 Cross product with unit vectors

In the last example we showed that \( \hat{i} \times \hat{j} = \hat{k} \). We can also show that

\[
\hat{i} \times \hat{i} = \hat{0} \quad \hat{i} \times \hat{j} = \hat{k} \quad \hat{i} \times \hat{k} = -\hat{j}
\]

(1.55)

\[
\hat{j} \times \hat{i} = -\hat{k} \quad \hat{j} \times \hat{j} = \hat{0} \quad \hat{j} \times \hat{k} = \hat{i}
\]

(1.56)

\[
\hat{k} \times \hat{i} = \hat{j} \quad \hat{k} \times \hat{j} = -\hat{i} \quad \hat{k} \times \hat{k} = \hat{0}
\]

(1.57)
Let \( \mathbf{a} = a_x \mathbf{i} + a_y \mathbf{j} + a_z \mathbf{k} \) and \( \mathbf{b} = b_x \mathbf{i} + b_y \mathbf{j} + b_z \mathbf{k} \). Then we can compute the cross product by multiplying these directly:

\[
\mathbf{a} \times \mathbf{b} = (a_x \mathbf{i} + a_y \mathbf{j} + a_z \mathbf{k}) \times (b_x \mathbf{i} + b_y \mathbf{j} + b_z \mathbf{k})
\]

\[
= a_x b_y (\mathbf{i} \times \mathbf{j}) + a_x b_z (\mathbf{i} \times \mathbf{k}) + a_y b_x (\mathbf{j} \times \mathbf{i}) + a_y b_z (\mathbf{j} \times \mathbf{k}) + a_z b_x (\mathbf{k} \times \mathbf{i}) + a_z b_y (\mathbf{k} \times \mathbf{j})
\]

Collecting terms we get

\[
\mathbf{a} \times \mathbf{b} = (a_y b_z - a_z b_y) \mathbf{i} + (a_z b_x - a_x b_z) \mathbf{j} + (a_x b_y - a_y b_x) \mathbf{k}
\]

### 1.5 Linear systems

Something that we will have to do regularly is solve systems of linear equations. For example:

\[
\begin{align*}
2x + 3y + z &= 4 \\
4x + 5y + z &= 2 \\
4y + 3z &= 2
\end{align*}
\]

How do we solve this system?

How can we write this set of equations using matrices and vectors?

\[
\begin{bmatrix}
2 & 3 & 1 \\
4 & 5 & 1 \\
0 & 4 & 3
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
= \begin{bmatrix}
4 \\
2 \\
2
\end{bmatrix}
\]

where \( A \) is called the coefficient matrix, and \( \mathbf{x} \) is the "vector of unknowns."

### 1.5.1 Cramer's rule

Cramer’s rule is an easy way to solve linear systems of equations without too much work. Suppose we have this coefficient matrix (note: this introduces some slightly new notation)

\[
\begin{bmatrix}
p_x & q_x & r_x \\
p_y & q_y & r_y \\
p_z & q_z & r_z
\end{bmatrix}
\equiv \begin{bmatrix}
p & q & r
\end{bmatrix}
\]

so that we have the linear system

\[
\begin{bmatrix}
p & q & r
\end{bmatrix} \begin{bmatrix}
x \\
y \\
z
\end{bmatrix} = \begin{bmatrix}
b
\end{bmatrix}
\]

Then the solution is given by

\[
x_1 = \frac{\det [b \ b \ r]}{\det [p \ q \ r]} \quad x_2 = \frac{\det [p \ b \ r]}{\det [p \ q \ r]} \quad x_3 = \frac{\det [p \ q \ b]}{\det [p \ q \ r]}
\]

(Notice that this can be generalized to \( n \)-dimensional systems. But in this class we’ll pretty much stick to 2 and 3.)
Let's solve the system we had before:

\[
\begin{bmatrix}
2 & 3 & 1 \\
4 & 5 & 1 \\
0 & 4 & 3 \\
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z \\
\end{bmatrix} =
\begin{bmatrix}
4 \\
2 \\
2 \\
\end{bmatrix}
\]  
(1.70)

To save some work:

\[ \text{det}(A) = (2)(3)(5) + (3)(1)(0) + (1)(4)(4) - (2)(1)(4) - (3)(4)(3)(1)(5)(0) = 30 + 16 - 8 - 36 = 2 \]  
(1.71)

Now we only have to compute three more determinants:

\[
\begin{bmatrix}
4 & 3 & 1 \\
2 & 5 & 1 \\
2 & 4 & 3 \\
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z \\
\end{bmatrix} = 60 + 6 + 8 - 16 - 18 - 10 = 30
\]  
(1.72)

\[
\begin{bmatrix}
4 & 4 & 1 \\
2 & 2 & 1 \\
0 & 2 & 3 \\
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z \\
\end{bmatrix} = 12 + 0 + 8 - 4 - 48 - 0 = -32
\]  
(1.73)

\[
\begin{bmatrix}
4 & 3 & 4 \\
2 & 4 & 2 \\
0 & 4 & 2 \\
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z \\
\end{bmatrix} = 20 + 0 + 64 - 16 - 24 - 0 = 44
\]  
(1.74)

So our solution is

\[
\begin{bmatrix}
x \\
y \\
z \\
\end{bmatrix} = \begin{bmatrix}
15 \\
-16 \\
22 \\
\end{bmatrix}
\]  
(1.75)

Notes:

- What happens when the determinant of the coefficient matrix is zero? The solution vector is infinite
  \[ \Rightarrow \text{A solution exists for the system if and only if the determinant of the coefficient matrix is nonzero.} \]
  - Always a good idea to check the determinant first. It will be useful in solving the system, and it may help you to find errors in your work.

- What if the coefficient matrix is not square? You can only solve the system with square matrices.
  \[ \Rightarrow \text{Number of solutions must equal number of unknowns.} \]

### 1.5.2 Proof of Cramer's rule

One of the nice things about Cramer’s rule is that it’s very easy to prove. Suppose we have the 3D system

\[ \text{det} \begin{bmatrix} p & q & r \\ x & y & z \end{bmatrix} = px + qy + rz = b \]  
(1.76)

Let us consider the following determinant:

\[ \text{det} \begin{bmatrix} (px + qy + rz) & q & r \\ p & q & r \end{bmatrix} \text{ which is equal to } \text{det} \begin{bmatrix} b & q & r \end{bmatrix} \]  
(1.77)

We know by the linearity of the determinant that we can write

\[ \text{det} \begin{bmatrix} (px + qy + rz) & q & r \end{bmatrix} = x \text{det} \begin{bmatrix} p & q & r \end{bmatrix} + y \text{det} \begin{bmatrix} q & q & r \end{bmatrix} + z \text{det} \begin{bmatrix} r & q & r \end{bmatrix} \]  
(1.78)

The latter two terms have a repeated column, so they must be zero. So we have

\[ x \text{det} \begin{bmatrix} p & q & r \end{bmatrix} = \text{det} \begin{bmatrix} b & q & r \end{bmatrix} \]  
(1.79)

\[ x = \frac{\text{det} \begin{bmatrix} b & q & r \end{bmatrix}}{\text{det} \begin{bmatrix} p & q & r \end{bmatrix}} \]  
(1.80)

and so on for \( y \) and \( z \).
2 Point Equilibrium

2.1 Free body diagrams

Definition 2.1. A free body diagram (FBD) for an object is a simple diagram of an object’s geometry, constraints, forces, and moments.

2.1.1 For a point

Consider the following complex system:

For this problem, the FBD is the simplest figure possible that contains all the information we need to know to solve the system.

Notes:

• Drawing a FBD establishes a convention for your problem. (e.g. assume a member is in tension.)
• More than one FBD may be necessary for a complex system

2.1.2 Diagram cutting

It is frequently useful to “cut” a diagram. It is easy to get confused about how to draw the forces. Consider the following very simple system:

We wish to draw FBDs for each, so we “cut” the diagram in two. To account for the effect of the “other half” we must add a force acting at the boundary.

Important point: the sum of the forces at a cut from both sides must equal zero.
2.2 Force bearing members

2.2.1 Pivots

Pivots denoted like this:

Attributes:
- constrained position in x,y – sustained force in x,y
- free rotation

2.2.2 Rollers

Rollers denoted like this:

Attributes:
- constrained position parallel to surface – sustained force parallel to surface
- free position along surface – no force along surface
- free rotation

2.2.3 Two force members (2FM)

Consider the following system:

Notes:
- 2FM cannot have forces acting midway, only at ends
- forces are co-linear, equal, and opposite
- forces are parallel with member (or else member would spin)
- use convention of tension = positive. always make sure to be consistent!
  ⇒ if beam ends up being in compression, the tension will be negative.
2.2.4 Cables

Cables have the following properties:

- as 2FM (tension only)
- weightless
- constant tension

2.2.5 Pulleys

Pulleys look like this. (Draw a free body diagram)

Forces acting on pulleys:

- cables (tension is constant)
- reaction force from fixed mount
- external forces

2.2.6 Springs

(Linear) springs are 2FMs with variable length

- Tension on a spring is given by
  \[ t = k \Delta L = k(L_f - L_i) \]  
  (2.1)
- Spring constant: units \( N/m \)
- Tension or compression (but usually tension unless indicated by a guide)

2.3 Governing equations

2.3.1 Equilibrium

For point equilibrium:

\[ \sum f = 0 \]  
(2.2)

(remember: this gives you 2 equations in 2D and 3 equations in 3D)
2.3.2 Geometric
Equations that describe constrained geometry: e.g.
- fixed cable length
- roller on specified track

2.3.3 Constitutive
Equations that describe a member’s response to a loading: e.g.
- spring stiffness equation
- noncompressive forces on a 2-force member
- maximum allowable tension

2.4 Problem methodology
1. Draw free body diagram
2. Identify unknown quantities
3. Write down equilibrium, geometric, and constitutive equations
4. Solve for unknowns
5. Test the solution (“sanity check”)
6. Substitute numerical values

2.5 2D examples

Example 2.1
Consider the following cable-pulley-spring system.
- length of the cable is $\ell$
- unstretched length of the spring is $h$

Find the equilibrium position of the pulley and the tension in the cable.
1. Free body diagram:

2. Unknown quantites: $x$, $y$, $t_{\text{cable}}$

3. Equations:
   (a) Equilibrium
   \[ f_{\text{cable1}} + f_{\text{cable2}} + f_{\text{spring}} + w = 0 \]  
   \[ (2.3) \]
   (b) Geometric
   \[ \ell_1 + \ell_2 = \sqrt{x^2+y^2} + \sqrt{(L-x)^2+y^2} = \ell \]  
   \[ (2.4) \]
   (c) Constitutive
   \[ t_{\text{spring}} = k((y+h) - h) = ky \]  
   \[ (2.5) \]

4. Solve for unknowns:
   x component of equilibrium:
   \[ \frac{t_{\text{cable}}}{\sqrt{x^2+y^2}}(-x) + \frac{t_{\text{cable}}}{\sqrt{(L-x)^2+y^2}}(L-x) + 0 + 0 = 0 \]  
   \[ (2.6) \]
   \[ \frac{L-x}{\sqrt{(L-x)^2+y^2}} = \frac{x}{\sqrt{x^2+y^2}} \]  
   \[ (2.7) \]
   guess: does $x = L/2$ work?
   \[ \frac{L/2}{\sqrt{(L/2)^2+y^2}} = \frac{(L/2)}{\sqrt{(L/2)^2+y^2}} \rightarrow \text{yes!} \rightarrow x = \frac{L}{2} \]  
   \[ (2.8) \]
now, solve for \( y \) → use geometry equation

\[
\sqrt{\left(\frac{L}{2}\right)^2 + y^2} + \sqrt{\left(\frac{L}{2}\right)^2 + y^2} = \ell
\]

(2.9)

\[
2\sqrt{\frac{L^2}{4} + y^2} = \ell
\]

(2.10)

\[
\frac{L^2}{4} + y^2 = \frac{\ell^2}{4}
\]

(2.11)

\[
y^2 = \frac{\ell^2 - L^2}{4}
\]

(2.12)

\[
y = \sqrt{\frac{\ell^2 - L^2}{2}} \rightarrow \text{units}\?\]  

(2.13)

(2.14)

what do we want now? we can get \( t_{\text{spring}} \) easily, so let’s do it!

\[
t_{\text{spring}} = ky = \frac{k\sqrt{\ell^2 - L^2}}{2}
\]

(2.15)

what equations have we used? all but the \( y \) component of equilibrium.
what unknowns do we still need? just one: the tension of the cables.

\[
\sum F_y = \frac{t_{\text{cable}}y}{\sqrt{\frac{L^2}{4} + y^2}} + \frac{t_{\text{cable}}y}{\sqrt{\frac{L^2}{4} + y^2}} + \frac{k\sqrt{\ell^2 - L^2}}{2} - w = 0
\]

(2.16)

\[
= \frac{2t_{\text{cable}}y}{\ell/2} + \frac{k\sqrt{\ell^2 - L^2}}{2} = w
\]

(2.17)

\[
= \frac{2t_{\text{cable}}\sqrt{\ell^2 - L}}{\ell} + \frac{k\sqrt{\ell^2 - L^2}}{2} = w
\]

(2.18)

\[
t_{\text{cable}} = \frac{\ell}{2} \left( \frac{w}{\sqrt{\ell^2 - L}} - \frac{k}{2} \right) \rightarrow \text{units}\?\]  

(2.19)

5. Test solution

\[
x = \frac{L}{2}
\]

(2.20)

• intuition? ✓

\[
y = \frac{\sqrt{\ell^2 - L^2}}{2}
\]

(2.21)

• what if \( \ell < L \)? is this physically possible?

\[
t_{\text{cable}} = \frac{\ell}{2} \left( \frac{w}{\sqrt{\ell^2 - L}} - \frac{k}{2} \right)
\]

(2.22)

• what if \( \ell = L \)? does it make sense for the tension to be infinite?
• when might \( t_{\text{cable}} \) is negative? would our solution still be valid? does this make sense?

6. Now, (if provided), we plug in numbers.
Example 2.2

A mass of weight \( w \) was attached to the system as shown:

Before the weight was attached, both springs were pre-stretched by a distance \( y \). Given values for \( k, L, y, \) and \( \theta \), what is the value of \( w \)?

1. Free body diagram:
   The unit vectors for the two cables can be found easily:
   
   \[
   \mathbf{f}_1 = t \begin{bmatrix} -\cos(\theta) \\ \sin(\theta) \end{bmatrix}, \quad \mathbf{f}_2 = t \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 0 \\ -w \end{bmatrix}
   \]

2. Identify unknown quantity: \( w \)

3. Governing equations
   (a) Equilibrium
   
   \[
   \mathbf{f}_1 + \mathbf{f}_2 + \mathbf{w} = \mathbf{0}
   \]  
   (2.23)

   (b) Geometric
   
   \[
   \frac{L}{2} = \frac{\ell}{2} \cos \theta \quad \Rightarrow \quad \ell = \frac{L}{\cos \theta}
   \]  
   (2.24)

   (c) Constitutive: If not prestretched:
   
   \[
   t = k \Delta x = k \left( \frac{\ell}{2} - \frac{L}{2} \right)
   \]  
   (2.25)

   But this was prestretched, so we have
   
   \[
   t = k \left( y + \frac{\ell}{2} - \frac{L}{2} \right)
   \]  
   (2.26)
4. Solve for unknowns: use the y component of the equilibrium equation

\[ t \sin(\theta) + t \sin(\theta) - w = 0 \]  

(2.27)

\[ w = k \left( 2y + \ell - L \right) \sin(\theta) = k \left( 2y + \frac{L}{\cos \theta} - L \right) \sin(\theta) \]  

(2.28)

5. Sanity checks:
   (a) units? ✓
   (b) what if \( \theta = 0 \)? then \( w = 0 \), which makes sense.

6. Now plug in values:

\[ k = 30 \frac{N}{m} \quad L = 4m \quad \theta = 30^\circ \quad y = 1m \]  

(2.29)

So

\[ w = (30N/m) \left( 2(1m) + \frac{4m}{\cos 30^\circ} - 4m \right) \sin(30^\circ) = 39.28N \]  

(2.30)

Example 2.3

Consider the following system:

Compute the tension in each cable in terms of \( w, \theta_1, \theta_2 \)

If the cables break under a tension of 100N, find the maximum load allowable if \( \theta_1 = 30.00^\circ, \theta_2 = 51.15^\circ \)

Before drawing the FBD, let's compute the unit vectors:

\[ \mathbf{n}_1 = \begin{bmatrix} \cos \theta_1 \\ \sin \theta_1 \end{bmatrix} \quad \mathbf{n}_2 = \pm \begin{bmatrix} -\cos \theta_2 \\ \sin \theta_2 \end{bmatrix} \quad \mathbf{n}_3 = \pm \begin{bmatrix} -1 \\ 0 \end{bmatrix} \quad \mathbf{n}_4 = \begin{bmatrix} \cos \theta_1 \\ \sin \theta_1 \end{bmatrix} \]  

(2.31)

Do they have magnitude of unity? Yes, \( \sin/\cos \) unit vectors in 2d always will.

1. Free body diagram:
   We need two of them
2. Identify unknowns: \( t_1, t_2, t_3, t_4 \)
3. Write governing equations:
   - The equilibrium equations are:
     \[
     \sum F = f_1 + f_2 + w = t_1 \begin{bmatrix} \cos \theta_1 \\ \sin \theta_1 \end{bmatrix} + t_2 \begin{bmatrix} -\cos \theta_2 \\ \sin \theta_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -w \end{bmatrix} = \mathbf{0} \tag{2.32}
     \sum F = f_2 + f_3 + f_4 = t_2 \begin{bmatrix} \cos \theta_2 \\ -\sin \theta_2 \end{bmatrix} + t_3 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + t_4 \begin{bmatrix} \cos \theta_1 \\ \sin \theta_1 \end{bmatrix} = \mathbf{0} \tag{2.33}
     \]
   - Good news! We have four equations and four unknowns, so now we can solve the problem.
     - Constitutive: \( t_i \leq 100N \)
4. Solve for unknowns: start with the first equilibrium equation, write in matrix notation
   \[
   \begin{bmatrix} \cos \theta_1 & -\cos \theta_2 \\ \sin \theta_1 & \sin \theta_2 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} 0 \\ w \end{bmatrix} \tag{2.34}
   \]
   - Check the determinant:
     \[
     \det \begin{bmatrix} \cos \theta_1 & -\cos \theta_2 \\ \sin \theta_1 & \sin \theta_2 \end{bmatrix} = \cos \theta_1 \sin \theta_2 + \cos \theta_2 \sin \theta_1 = \sin(\theta_1 + \theta_2) \tag{2.35}
     \]
   - When would this be equal to zero? For instance, when \( \theta_1 = \theta_2 = 0, \theta_1 = \theta_2 = 90^\circ \). For now, we'll assume that those cases don't happen and that the system is solvable.
     - \( t_1 = \frac{1}{\sin(\theta_1 + \theta_2)} \det \begin{bmatrix} 0 & -\cos \theta_2 \\ w & \sin \theta_2 \end{bmatrix} = \frac{w \cos \theta_2}{\sin(\theta_1 + \theta_2)} \tag{2.36} \)
     - \( t_2 = \frac{1}{\sin(\theta_1 + \theta_2)} \det \begin{bmatrix} \cos \theta_1 & 0 \\ \sin \theta_1 & w \end{bmatrix} = \frac{w \cos \theta_1}{\sin(\theta_1 + \theta_2)} \tag{2.37} \)
   - Great, we've solved for \( t_1 \) and \( t_2 \). Now we need the other two, so let's use the second equilibrium equation to solve in terms of \( t_2 \)
     - \( t_3 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + t_4 \begin{bmatrix} \cos \theta_1 \\ \sin \theta_1 \end{bmatrix} = t_2 \begin{bmatrix} -\cos \theta_2 \\ \sin \theta_2 \end{bmatrix} \tag{2.38} \)
     - \( \begin{bmatrix} -1 \\ \cos \theta_1 \end{bmatrix} \begin{bmatrix} t_3 \\ t_4 \end{bmatrix} = \begin{bmatrix} -t_2 \cos \theta_2 \\ t_2 \sin \theta_2 \end{bmatrix} \tag{2.39} \)
   - Check the determinant:
     - \( \det \begin{bmatrix} -1 & \cos \theta_1 \\ 0 & \sin \theta_1 \end{bmatrix} = -\sin \theta_1 \tag{2.40} \)
Does this make sense? Yes, because the problem would be poorly defined if the third and fourth cables are parallel. Assume that this is not the case, and solve for $t_3$, $t_4$

\[
\begin{align*}
    t_3 &= \frac{1}{-\sin \theta_1} \det \begin{bmatrix} -t_2 \cos \theta_2 & \cos \theta_1 \\ t_2 \sin \theta_2 & \sin \theta_1 \end{bmatrix} = \frac{t_2(-\cos \theta_2 \sin \theta_1 - \cos \theta_1 \sin \theta_2)}{-\sin \theta_1} = \frac{t_2 \sin(\theta_1 + \theta_2)}{\sin \theta_1} \\
    t_4 &= \frac{1}{-\sin \theta_1} \det \begin{bmatrix} -1 & -t_2 \cos \theta_1 \\ 0 & t_2 \sin \theta_2 \end{bmatrix} = \frac{-t_2 \sin \theta_2}{-\sin \theta_1} = \frac{t_2 \sin \theta_2}{\sin \theta_1} 
\end{align*}
\]

Substituting for $t_2$ gives the final result:

\[
\begin{align*}
    t_1 &= \frac{w \cos \theta_2}{\sin(\theta_1 + \theta_2)} \\
    t_2 &= \frac{w \cos \theta_1}{\sin(\theta_1 + \theta_2)} \\
    t_3 &= \frac{w}{\tan \theta_1} \\
    t_4 &= \frac{w \sin \theta_2}{\sin(\theta_1 + \theta_2) \tan \theta_1}
\end{align*}
\]

5. Sanity check:
   - units? ✓
   - $\theta_1 = \theta_2 = 0^\circ$ makes sense—that problem would be poorly defined.

6. Plug in values: $\theta_1 = 30.00^\circ$, $\theta_2 = 53.15^\circ$: then

\[
\begin{align*}
    t_1 &= 0.6040w \\
    t_2 &= 0.8723w \\
    t_3 &= 1.732w \\
    t_4 &= 1.396w
\end{align*}
\]

Apparently, for this case, cable 3 carries the majority of the weight, so it will fail first. Substituting $t_3 = 100N$ and solving for $w$ gives

\[
w = \frac{100N}{1.732} = 57.74N
\]

2.6 3D force-bearing members

2.6.1 2D members
All of the 3d force-bearing members carry over (2FM, pulleys, etc.)

2.6.2 3D pivot
The 3d pivot is the 3d analog of the 2d pivot:

- Restricted motion in $x$, $y$, $z$, sustained force
- Free rotation in $x$, $y$, $z$

2.7 3D free body diagrams
- especially important to define coordinate system!
- lots of boxes can be helpful
2.8 3D equilibrium equation

\[ \sum f = 0 \]  
\[ \sum f_x = 0 \]  
\[ \sum f_y = 0 \]  
\[ \sum f_z = 0 \]

In 3D this gives us 3 equations:

\[ \sum f_x = 0 \]  
\[ \sum f_y = 0 \]  
\[ \sum f_z = 0 \]

2.9 3D examples

**Example 2.4**

A box of weight \( w \) is suspended using three cables as shown.

Determine the tensions in the cables in terms of \( w \). Which cable carries the most weight?

1. Free body diagram:
   It is very important to define a coordinate system. Then the free body diagram is:

\[ f_1 = t_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \]

\[ f_2 = \frac{t_2}{\sqrt{49}} \begin{bmatrix} -6 \\ -2 \\ 0 \end{bmatrix} = t_2 \begin{bmatrix} -6/7 \\ -2/7 \\ 0 \end{bmatrix} \]

\[ f_3 = \frac{t_3}{\sqrt{49}} \begin{bmatrix} -6 \\ 2 \\ 0 \end{bmatrix} = t_3 \begin{bmatrix} -6/7 \\ 2/7 \\ 0 \end{bmatrix} \]

2. Identify unknowns:
   We want to solve for \( t_1, t_2, t_3 \) (3 unknowns)

3. Write equations:
   \[ \sum F = 0 \] (3 equations)

   Great! We are all set, all we need to do is solve equilibrium.
4. Solve for unknowns

\[ \mathbf{f}_1 + \mathbf{f}_2 + \mathbf{f}_3 + \mathbf{w} = \mathbf{0} \]  \hspace{1cm} (2.49)

\[
\begin{bmatrix}
1 \\
0 \\
t_1
\end{bmatrix} +
\begin{bmatrix}
-6/7 \\
-2/7 \\
t_2
\end{bmatrix} +
\begin{bmatrix}
-6/7 \\
3/7 \\
t_3
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
\mathbf{w}
\end{bmatrix} \hspace{1cm} (2.50)
\]

\[
\begin{bmatrix}
7 \\
0 \\
0
\end{bmatrix} -
\begin{bmatrix}
-6 \\
-2 \\
3
\end{bmatrix} +
\begin{bmatrix}
-6 \\
3 \\
2
\end{bmatrix} t_1 +
\begin{bmatrix}
-6 \\
-2 \\
3
\end{bmatrix} t_2 +
\begin{bmatrix}
-6 \\
3 \\
2
\end{bmatrix} t_3 =
\begin{bmatrix}
0 \\
0 \\
\mathbf{w}
\end{bmatrix} \hspace{1cm} (2.51)
\]

As usual, compute the determinant to make sure we can solve the thing:

\[
\det \begin{bmatrix}
7 & -6 & -6 \\
0 & -2 & 3 \\
0 & 3 & 2
\end{bmatrix} = 7(-4 - 9) = -91 \hspace{1cm} (2.52)
\]

so it’s solvable. Now we can easily compute the coefficients:

\[
t_1 = -\frac{1}{91} \det \begin{bmatrix}
0 & -6 & -6 \\
0 & -2 & 3 \\
7w & 3 & 2
\end{bmatrix} = -\frac{1}{91} (7w(-18 - 12)) = \frac{210}{91} w = \frac{30}{13} w \hspace{1cm} (2.53)
\]

\[
t_2 = -\frac{1}{91} \det \begin{bmatrix}
7 & 0 & -6 \\
0 & 0 & 3 \\
0 & 7w & 2
\end{bmatrix} = -\frac{1}{91} (-147w) = \frac{21}{13} w \hspace{1cm} (2.54)
\]

\[
t_3 = -\frac{1}{91} \det \begin{bmatrix}
7 & -6 & -6 \\
0 & -2 & 3 \\
0 & 0 & 7w
\end{bmatrix} = -\frac{1}{91} (-98w) = \frac{14}{13} w \hspace{1cm} (2.55)
\]

5. Sanity check

- units? ✓
- negative value? negative weight (pushing up) ✓

(the greatest tension is in cable 1).
Example 2.5

Consider the following cable-strut system:

Compute the tension (or compression) in the cable and the struts in terms of \(L, W, H, y, w\).

1. Draw a free body diagram

\[
f_{\text{cable}} = \frac{t_{\text{cable}}}{\ell_{\text{cable}}} \begin{bmatrix} -L \\ -y \\ -H \end{bmatrix}
\]

\[
f_{\text{strut}1} = \frac{t_{\text{strut}}}{\ell_{\text{strut}}} \begin{bmatrix} L \\ -y \\ -H \end{bmatrix}
\]

\[
f_{\text{strut}2} = \frac{t_{\text{strut}}}{\ell_{\text{strut}}} \begin{bmatrix} -L \\ -y \\ -H \end{bmatrix}
\]

2. Identify unknowns: \(t_{\text{strut}1}, t_{\text{strut}2}, t_{\text{cable}}\)

3. Write the equations:

\[
\sum f = f_{\text{strut}1} + f_{\text{strut}2} + f_{\text{cable}} + w = 0 \quad (2.56)
\]

4. Solve. To make things easier, let’s define:

\[
\ell_{\text{strut}} \equiv \sqrt{L^2 + y^2 + H^2} \quad \ell_{\text{cable}} \equiv \sqrt{(W + y)^2 + H^2} \quad (2.57)
\]

Now, write equilibrium in matrix form:

\[
\begin{bmatrix} L/\ell_{\text{strut}} & -L/\ell_{\text{strut}} & 0 \\ -y/\ell_{\text{strut}} & -y/\ell_{\text{strut}} & -(y + W)/\ell_{\text{cable}} \\ -H/\ell_{\text{strut}} & -H/\ell_{\text{strut}} & -H/\ell_{\text{cable}} \end{bmatrix} \begin{bmatrix} t_{\text{strut}1} \\ t_{\text{strut}2} \\ t_{\text{cable}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ w \end{bmatrix} \quad (2.58)
\]

Check the determinant:

\[
det = \frac{LyH}{\ell_{\text{strut}}^2 \ell_{\text{cable}}} - \frac{L(y + W)H}{\ell_{\text{strut}}^2 \ell_{\text{cable}}} - \frac{L(y + W)H}{\ell_{\text{strut}}^2 \ell_{\text{cable}}} = \frac{2LHW}{\ell_{\text{strut}}^2 \ell_{\text{cable}}} \quad (2.59)
\]
When would it be zero? If \( L = 0, H = 0, W = 0 \). We'll assume that this is not the case.

\[
t_{\text{strut}} = -\frac{\ell_{\text{strut}}^2 \ell_{\text{cable}}}{2LHW} \begin{bmatrix} 0 & -L/\ell_{\text{strut}} & 0 \\ 0 & -y/\ell_{\text{strut}} & -(y + W)/\ell_{\text{cable}} \\ -w & -H/\ell_{\text{strut}} & -w/\ell_{\text{cable}} \end{bmatrix} = -\frac{\ell_{\text{strut}}^2 \ell_{\text{cable}}}{2LHW} \frac{(L(y + W)w)}{\ell_{\text{strut}} \ell_{\text{cable}}} \tag{2.60}
\]

\[
t_{\text{strut}} = -\frac{w(y + W)\sqrt{L^2 + y^2 + H^2}}{2HW} \tag{2.61}
\]

\[
t_{\text{strut}} = ... = -\frac{(y + W)\sqrt{L^2 + y^2 + H^2}}{2HW} = t_{\text{strut1}} \tag{2.62}
\]

\[
t_{\text{cable}} = -\frac{\ell_{\text{strut}}^2 \ell_{\text{cable}}}{2LHW} \begin{bmatrix} L/\ell_{\text{strut}} & -L/\ell_{\text{strut}} & 0 \\ -y/\ell_{\text{strut}} & -y/\ell_{\text{strut}} & 0 \\ -H/\ell_{\text{strut}} & -H/\ell_{\text{strut}} & w \end{bmatrix} = -\frac{\ell_{\text{strut}}^2 \ell_{\text{cable}}}{2LHW} \frac{-Lyw + Lyw}{\ell_{\text{strut}}} \tag{2.63}
\]

\[
t_{\text{cable}} = \frac{w \sqrt{(y + W)^2 + H^2}}{HW} \tag{2.64}
\]

5. Sanity check:
   - units?
     \[ t_{\text{strut1,2}} = \text{[force]} \begin{bmatrix} \text{[length]} & \text{[length]} \\ \text{[length]} & \text{[length]} \end{bmatrix} = \text{force ✓} \quad t_{\text{cable}} = \text{[force]} \begin{bmatrix} \text{[length]} & \text{[length]} \\ \text{[length]} & \text{[length]} \end{bmatrix} \tag{2.65} \]
   - undefined value? yes, when \( H, W = 0 \) as we expected
   - sign? assuming values in the picture, \( t_{\text{strut1,2}} < 0, t_{\text{cable}} > 0 \) as expected.
   - what if \( y < 0 \)? then all the tensions are compressive.Does this make sense?
   - zeros? what if \( y = 0 \)? then \( t_{\text{cable}} = 0 \). does this make sense?
   - what if \( y < -W \)? then the cable is in compression but the struts are in tension. Does this make sense?

6. Substitute values. No values given.

### 2.10 Static indeterminacy

As we saw last time, we occasionally encounter cases where our static system is unsolvable, yet our physical intuition tells us that the system is well-defined. For instance, consider the following system:

\[
\begin{align*}
\mathbf{f}_1 &= \frac{h}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \\
\mathbf{f}_2 &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
\mathbf{f}_3 &= \frac{h}{\sqrt{3}} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \\
\mathbf{w} &= \begin{bmatrix} 0 \\ 0 \\ -w \end{bmatrix}
\end{align*}
\]

The forces are already given to us so we’ll dispense with the FBD for now, and simply write the equilibrium equations.
for the node.

\[ f_1 + f_2 + f_3 + w = 0 \]  \hspace{1cm} (2.66)

\[
\begin{bmatrix}
\frac{1}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{3}} \\
-\frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} \\
\frac{1}{\sqrt{3}} & 1 & \frac{1}{\sqrt{3}}
\end{bmatrix}
\begin{bmatrix}
f_1 \\
f_2 \\
f_3
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
w
\end{bmatrix}
\]  \hspace{1cm} (2.67)

As usual, we end up with a linear system. Our first step to solving linear systems, as always, is to compute the determinant:

\[ \text{det} = 0 + 0 + \frac{1}{3} - \frac{1}{3} - 0 - 0 = 0 \]  \hspace{1cm} (2.68)

The determinant is zero, which means that we will be unsuccessful in solving the system. Yet, looking at the picture, we have no reason to suspect that it would not be in static equilibrium. In this case, there are a couple of reasons why we cannot find a solution:

(1) There are **multiple solutions** to this problem. It could be that the load is carried entirely by the center cable, or split evenly between the other two cables, or some combination of the above. In other words, there are an infinite number of solutions, so we cannot use our equations of statics to find it. (When we get to mechanics of materials, we will treat the cables as springs. This gives us the additional equations we need to solve the system.)

(2) We note that even though the problem is drawn in 3D, it is **effectively 2D** – that is, it would be 2D if we rotated our coordinate system by 45°. In that case, we would end up with only two equations for three unknowns.

Systems like this are called **statically indeterminant**, and it is important to learn to recognize these systems. Note: a helpful rule of thumb for finding static determinacy (aside from writing out all the equations) is to conduct the thought experiment of removing a member and seeing if the structure would collapse. If the structure always collapses upon the removal of any member, it is generally statically determinant.

### 3 Moments

Up until now, we have only been able to look at systems where all of the force pass through a single point. This is useful for a number of applications (e.g. trusses, as we'll see in a few weeks) but it limits us to very simple systems. Here, we will introduce another balance law that allows us to do equilibrium easily on complex bodies.

#### 3.1 Force-bearing members

Before talking moments, let us introduce a number of objects that we will use when doing rigid body equilibrium.

#### 3.1.1 Rigid bodies

Rigid bodies are bodies that can sustain arbitrary forces and arbitrary moments (we'll explain those shortly.) We draw rigid bodies in the following way:
We note that there are no restrictions on the location or direction of the forces, other than that they must all sum to zero. We also note that rigid bodies are not allowed to deform: for instance, we could not have a rigid body that contains a hinge. Rigid bodies are much more “sophisticated” than our previous members, 2FMVs, which could only sustain two colinear forces.

### 3.1.2 Rigid joints

Previously we looked at pivots, which sustain a 2D force and restrict 2D motion, but allow rotation. Now, we will consider rigid joints that we will draw as:

- Like with pivots, rigid joints restrict motion in 2D which mean that they sustain a force in 2D.
- Unlike pivots, rigid joints restrict rotation as well. This means that they must sustain a “force” that opposes the rotation: this is a moment.

### 3.1.3 Hinges

Hinges have similarities to both pivots and rigid joints. We will draw hinges as:

- As with pivots and rigid joints, we see that hinges restrict motion in all directions. This means that it sustains a 3d force.
- We see that there is a free rotation allowed about the hinge axis. That means that there is no “force” (or moment) opposing the rotation in that direction.
- However, we see that the hinge does not allow twisting or rotation about the other two axes. Therefore, it must sustain a moment about the other two. Here, we start to see the need to express a “rotational force,” and this example shows that it must be three-dimensional.

### 3.2 Moments

In the previous section we considered only systems of “concurrent forces,” that is systems, where all force vectors passed through a common point. Because they all pass through a single point, there can be no rotation. Now, we will consider more complex systems in which rotational degrees of freedom must be taken into account.

#### 3.2.1 Motivation

Previously we talked about 2FMVs and argued that the forces had to be equal and opposite, and colinear with the length of the beam.

In the second case, the sum of the forces equals zero but the **sum of the moments** does not. Lack of force equilibrium causes linear acceleration, lack of moment equilibrium causes angular acceleration.
3.2.2 Definition of a moment

**Definition 3.1.** A moment is the conjugate force to rotational motion. The moment of a force about a point \( p \) is the cross product of the distance vector (the vector pointing from \( p \) to the point of application of the force) with the force vector.

In vector notation:

\[
M = r \times f
\]

where \( r \) is the vector pointing from the reference point to the point of application of the force, and is in units of distance. A couple of things to note:

1. We take the direction of \( r \) as convention: the arrow should always be at the same point as the force.
2. The cross product is *anticommutative*, that is \( a \times b = -b \times a \). Therefore, it is very important that we stick with the convention of \( M = r \times f \), not the other way around.
3. We note that moment is a vector quantity and is only defined in 3D. (For 2D systems, the moment vector always points in the \( z \) direction only.)
4. units: \( n \cdot M \)

NB: this is technically the same units as energy (Joules), but we do not refer to them as Joules, simply Newton-meters. The reason for this is that

\[
[moment] \times [\text{angle turned}] = [\text{energy}]
\]

but angles are in radians which are unitless. You can think of moments as being in units of Newton-meters/radian.

### Example 3.1

Consider the following system:

\[ f = \begin{bmatrix} -1N \\ 1N \end{bmatrix} \]

\( d = (1m, 1m) \)

\( c = (1m, 2m) \)

\( a = (0m, 0m) \)

\( b = (2m, 0m) \)

Compute the moment of \( f \) about \( a, b, c \).

(a) compute distance vector:

\[
r_{ad} = \begin{bmatrix} 1m - 0m \\ 1m - 0m \end{bmatrix} = \begin{bmatrix} 1m \\ 1m \end{bmatrix}
\]

compute cross product:

\[
M_{ad} = r_{ad} = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1m & 1m & 0 \\ -1N & 1N & 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1Nm + 1Nm \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2Nm \end{bmatrix}
\]

Do we expect a positive value?

(b) compute distance vector

\[
r_{bd} = \begin{bmatrix} 1m \\ 1m \end{bmatrix} - \begin{bmatrix} 2m \\ 0m \end{bmatrix} = \begin{bmatrix} -1m \\ 1m \end{bmatrix}
\]
then compute cross product
\[
\mathbf{M}_b = \mathbf{r}_{bd} = \begin{bmatrix}
\hat{i} \\
-1m \\
\hat{j} \\
1m \\
\hat{k} \\
0
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
-1Nm + 1Nm
\end{bmatrix} = \mathbf{0}
\] (3.6)

Why is the vector zero? do we expect this?

(c) compute distance vector
\[
\mathbf{r}_{cd} = \begin{bmatrix}
-1m \\
0m
\end{bmatrix}
\] (3.7)
then compute cross product
\[
\mathbf{M}_c = \mathbf{r}_{cd} = \begin{bmatrix}
\hat{i} \\
-1m \\
\hat{j} \\
0m \\
\hat{k} \\
0
\end{bmatrix} = \begin{bmatrix}
0m \\
0m \\
-1Nm
\end{bmatrix}
\] (3.8)

Note that the moment is negative. Why is this? Because of the right hand rule.

### 3.2.3 Polar form
Recall the identity:
\[
||\mathbf{M}|| = ||\mathbf{r} \times \mathbf{f}|| = ||\mathbf{r}|| ||\mathbf{f}|| \sin \theta
\] (3.9)

Let's check this with the above results:

Example 3.2

(a) \( ||\mathbf{r}_{ad}|| = \sqrt{1m^2 + 1m^2} = \sqrt{2}m \) and \( ||\mathbf{f}|| = \sqrt{1N^2 + 1N^2} = \sqrt{2}N \) and \( \sin \theta = \sin 90^\circ = 1 \) so we have \( \sqrt{2}m(\sqrt{2}N)(1) = 2Nm \checkmark \)

(b) What is the angle? Looks like 180°, but it is actually 0°. So \( \sin \theta = 0 \) and we get 0. \( \checkmark \)

(c) How about the last one? \( ||\mathbf{r}_{cd}|| = 1m \) and \( \sin \theta = \cos 45^\circ = 1/\sqrt{2} \), so \( ||\mathbf{M}|| = (1m)(\sqrt{2}N)(1/\sqrt{2}) = 1Nm \checkmark \)

In the last example, notice that we **lost the information about the negativity of the moment**. When balancing moments using the polar form, one must keep track manually of which moments are positive and which are negative. How to do that? Right hand rule:

(Note: if you are consistent about defining your angles then you don't have to worry, but this is usually a pain. It's way easier to just use vectors.)

### 3.2.4 Pictorially
The moment vector is orthogonal to both the distance vector and the force vector. But the distance vector changes
• At the corner, \( \mathbf{M} \) is in the “twisting direction”
• At the base, \( \mathbf{M} \) has twisting and bending components
• The moment vector changes continuously throughout the body.
• The moment of a force along the line of action is always zero.
3.2.5 Line of application

Let us suppose we are taking the moment of a force \( \mathbf{f} \) about a point so that the distance vector is \( \mathbf{r} \).

\[
\mathbf{r} + \alpha \mathbf{f}
\]

Sometimes the location of the force makes finding a specific distance vector tedious. Let us consider a different distance vector that points from the same reference point to some arbitrary point along which \( \mathbf{f} \) acts. We can represent this by

\[
\hat{\mathbf{r}} = \mathbf{r} + \alpha \mathbf{f}
\]

where \( \alpha \) is some scalar multiplier (that, incidentally, should have units of distance/force). We can see that \( \hat{\mathbf{r}} \) always points to a point in the line along which \( \mathbf{f} \) acts, called the line of application. What is the resulting moment if we use this new distance vector?

\[
\hat{\mathbf{M}} = \hat{\mathbf{r}} \times \mathbf{f} = (\mathbf{r} + \alpha \mathbf{f}) \times \mathbf{f} = \mathbf{r} \times \mathbf{f} + \alpha \mathbf{f} \times \mathbf{f} = \mathbf{0}
\]

Therefore, we see that we can take our distance vector to be any vector that points from the reference point to the line of application of the force.

3.3 Moment about a specific axis

We can find the twisting moment about a specific direction \( \mathbf{d} \) using projection:

\[
M_{\text{twist}} = \frac{\mathbf{M} \cdot \mathbf{d}}{||\mathbf{d}||}
\]

Example 3.3

(a) In the above example, compute the moment vector at the base of the pipe:

\[
\mathbf{a} = (0, 0, 0)
\]

\[
\mathbf{b} = (0, 0, 1\, \text{m})
\]

\[
\mathbf{c} = (-5\, \text{m}, 5\, \text{m}, 1\, \text{m})
\]

\[
\mathbf{f} = \begin{bmatrix} -1 \, \text{N} \\ -1 \, \text{N} \\ 0 \end{bmatrix}
\]

\[
\mathbf{z}
\]

\[
x
\]

\[
y
\]

Compute moment about \( \mathbf{a} \): first compute

\[
\mathbf{r}_{ac} = \begin{bmatrix} -5 \, \text{m} \\ 5 \, \text{m} \\ 1 \, \text{m} \end{bmatrix}
\]

(3.13)
so the moment is
\[ \mathbf{M}_a = \mathbf{r}_{ac} \times \mathbf{f} = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ -5m & 5m & 1m \\ -1N & -1N & 0 \end{bmatrix} = \begin{bmatrix} 1Nm \\ -1Nm \\ 10Nm \end{bmatrix} \] (3.14)

(b) Compute the twisting moment along the pipe. Using the vector we got above, just take the projection. We will use the vector \( \mathbf{r}_{ab} \):
\[ \mathbf{r}_{ab} = \begin{bmatrix} 0 \\ 0 \\ 1m \end{bmatrix} \] (3.15)

so the projection is
\[ M_{\text{twist}} = \frac{\mathbf{M}_a \cdot \mathbf{r}_{ab}}{||\mathbf{r}_{ab}||} = \frac{(0m)(1Nm) + (0m)(1Nm) + (1m)(10Nm)}{1m} = 10Nm \] (3.16)

which, as we expect, is just the \( z \) component of the moment vector.

(c) Compute the bending moment at \( a \). We can write the twisting moment in vector form by multiplying by the unit vector:
\[ \mathbf{M}_{\text{twist}} = \mathbf{M}_{\text{twist}} \mathbf{n} = \frac{\mathbf{M}_{\text{twist}}}{||\mathbf{r}_{ab}||} \begin{bmatrix} 0 \\ 0 \\ 10Nm \end{bmatrix} \] (3.17)

Let us assume that we can write
\[ \mathbf{M} = \mathbf{M}_{\text{twist}} + \mathbf{M}_{\text{bend}} \] (3.18)

So then we can find \( \mathbf{M}_{\text{bend}} \) by subtraction:
\[ \mathbf{M}_{\text{bend}} = \mathbf{M} - \mathbf{M}_{\text{twist}} = \begin{bmatrix} 1Nm \\ -1Nm \\ 10Nm \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 10Nm \end{bmatrix} = \begin{bmatrix} 1Nm \\ -1Nm \\ 0 \end{bmatrix} \] (3.19)

[draw on picture]
This is the component of the moment that acts to bend the pipe at this point. (Notice how the vector is orthogonal to the direction of the pipe.)

### 3.4 Couple moments

Consider the following system where two equal and opposite forces are applied at points separated by the orthogonal distance vector \( \mathbf{d} \).

\[ \mathbf{M} = \mathbf{r}_0 \times -\mathbf{F} + (\mathbf{r}_0 + \mathbf{d}) \times \mathbf{F} = -\mathbf{r}_0 \times \mathbf{F} + \mathbf{r}_0 \times \mathbf{F} + \mathbf{d} \times \mathbf{F} = \mathbf{d} \times \mathbf{F} \] (3.20)
Notice that even though we were taking the moment about the origin, the distance from the origin \( r_0 \) was completely cancelled out. This is a result of the fact that the forces were equal and opposite. Whenever there are two equal and opposite forces that are separated by an offset vector, the result is a couple moment. There are some important things to note about couple moments:

1. If the moment is generated by forces with magnitude \( F \), and the orthogonal offset vector is \( d \) with magnitude \( d \), then the magnitude of the couple moment is

\[
||d \times F|| = ||d|| ||F|| \sin \theta = ||d|| ||F|| = dF
\]

2. The direction of the couple moment must be orthogonal to both forces, and is therefore orthogonal to the plane that they define. Use the right hand rule to remember the sign. Alternatively, if you are applying a couple moment to a screwdriver, the moment points in the direction that the screw is moving.

3. The location of a couple moment does not matter. As we saw above, because \( r_0 \) cancelled out, the couple moment has exactly the same effect about every point. (This is counterintuitive, but generally a pretty friendly fact. When doing rigid body pairs of forces like this produce a couple moment.

**Definition 3.2.** A couple moment is a pure moment that has no net force and can act at any point.

Pictorially:

![Pictorial representation of a couple moment](image)

**Example 3.4**

In order to install a screw, a screwdriver exerts a couple moment with magnitude \( M \) as shown.

To stabilize the part, two forces are applied at each end as shown. What magnitude of force should be applied so that the net moment is zero?

If we adopt a coordinate system such that the moment acts in the negative z direction, then the screwdriver moment is

\[
\begin{bmatrix}
0 \\
0 \\
- M
\end{bmatrix}
\]
The moment exerted by the forces is:

\[
\begin{align*}
\mathbf{r}_1 &= \begin{bmatrix} d/2 \\ h/2 \\ 0 \end{bmatrix} \\
\mathbf{r}_2 &= \begin{bmatrix} -d/2 \\ -h/2 \\ 0 \end{bmatrix} \\
\mathbf{F}_1 &= \begin{bmatrix} 0 \\ -F \\ 0 \end{bmatrix} \\
\mathbf{F}_2 &= \begin{bmatrix} 0 \\ F \\ 0 \end{bmatrix}
\end{align*}
\]

So for the moments to be equal:

\[
M = -\frac{dF}{2} - \frac{dF}{2} = -dF \implies F = -\frac{M}{d}
\]

Note that we never quantified where the moment had to be applied.

### 3.5 Reduction of forces and moments

We can show that each set of forces and moments can be combined to form a single effective force+moment combination acting at a particular point. For example, consider the following beam that is supporting a load \( w \) with two forces:

![Beam with forces](image)

We can write the supporting forces as two separate forces or as a force+a couple at a single point. It is often convenient to reduce a complex force system to a single force and a single moment. We will illustrate this with an example:

**Example 3.5**

Consider the following rigid body with forces and moments applied as shown.

![Rigid body with forces and moments](image)

Find the effective force+moment combination at points a and b.
(a) Let's begin by writing out all of the forces and distance vectors:

\[
\begin{align*}
f_1 &= F_1 \begin{bmatrix} \cos \theta_1 \\ \sin \theta_1 \\ 0 \end{bmatrix} \\
f_2 &= F_2 \begin{bmatrix} \cos \theta_2 \\ -\sin \theta_2 \\ 0 \end{bmatrix} \\
f_3 &= \begin{bmatrix} 0 \\ -F_3 \\ 0 \end{bmatrix} \\
r_1 &= \begin{bmatrix} L \\ H \\ 0 \end{bmatrix} \\
r_2 &= \begin{bmatrix} 2L \\ 0 \\ 0 \end{bmatrix} \\
r_3 &= \begin{bmatrix} L \\ 0 \\ 0 \end{bmatrix}
\end{align*}
\]

and the moment vector is

\[
\begin{bmatrix} 0 \\ 0 \\ M \end{bmatrix}
\]

The total force is just the sum of all of the forces:

\[
f_a = \begin{bmatrix} F_1 \cos \theta_1 + F_2 \cos \theta_2 \\ F_1 \sin \theta_1 - F_2 \sin \theta_2 - F_3 \\ 0 \end{bmatrix}
\]

And the total moment is the sum of all of the moments:

\[
M_a = (r_1 \times f_1) + (r_2 \times f_2) + (r_3 \times f_3) + M = \begin{bmatrix} 0 \\ 0 \\ F_1(L \sin \theta_1 - H \cos \theta_1) - 2F_2L \sin \theta_2 - F_3L + M \end{bmatrix}
\]

(b) What is the equivalent force vector acting at b? It is exactly the same: the total force vector is independent of location.

\[
f_a = f_b
\]

Now, we will compute the effective moment. First, of course, we compute distance vectors, remembering that the distance vector points to the force location:

\[
\begin{align*}
r_1 &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
r_2 &= \begin{bmatrix} L \\ -H \\ 0 \end{bmatrix} \\
r_3 &= \begin{bmatrix} 0 \\ -H \\ 0 \end{bmatrix}
\end{align*}
\]

What forces will contribute moments? Only \(f_2\): the first force is acting at the same point, and the third force is colinear, so the resultant moment is just

\[
M_b = (r_2 \times f_2) + M = \begin{bmatrix} 0 \\ 0 \\ F_2(-L \sin \theta_2 + H \cos \theta_2) + M \end{bmatrix}
\]

3.6 Distributed loads

Most of the examples we have looked at so far have involved “point loads” – that is, loads that are concentrated at a single point. They are convenient to work with mathematically, but in reality, loads are rarely that concentrated; they always have a finite width. In this section, we are going to look at how to work with loads that are distributed and change continuously in space.

3.6.1 Discrete case

We will begin by considering the discrete case and taking the limit. Consider the following beam loaded with boxes of varying height \(\{y_i\}\) and constant width \(\Delta x\)
Let each box have a specific weight (that is, weight per unit volume or area, should have units of \([\text{force}]/[\text{volume}]\) or \([\text{force}]/[\text{area}]\)) of \(w_i\), so that box \(i\) has weight \(w_i y_i \Delta x\). Each box exerts the following force vector:

\[
f_i = \begin{bmatrix} 0 \\ -w_i y_i \Delta x \end{bmatrix}
\]  \hspace{1cm} (3.35)

What is the effective (or total) force acting at the joint? To get this we just add all of them up:

\[
f = \begin{bmatrix} 0 \\ -w_1 y_1 \Delta x \end{bmatrix} + \begin{bmatrix} 0 \\ w_2 y_2 \Delta x \end{bmatrix} + \ldots = \sum_i \begin{bmatrix} 0 \\ w_i y_i \Delta x \end{bmatrix}
\]  \hspace{1cm} (3.36)

What is the effective \textit{moment} acting at the joint? For this, we need to compute a position vector for each box. The position vector for each one is just:

\[
r_i = \begin{bmatrix} x_i \\ 0 \\ 0 \end{bmatrix}
\]

so each box has a moment vector of

\[
M_i = r_i \times f_i = \begin{bmatrix} 0 \\ 0 \\ w_i y_i x_i \Delta x \end{bmatrix}
\]  \hspace{1cm} (3.37)

So, the total moment acting at the joint is:

\[
M = \begin{bmatrix} 0 \\ w_1 y_1 x_1 \Delta x \end{bmatrix} + \begin{bmatrix} 0 \\ w_2 y_2 x_2 \Delta x \end{bmatrix} + \ldots = \sum_i \begin{bmatrix} 0 \\ w_i y_i x_i \Delta x \end{bmatrix}
\]  \hspace{1cm} (3.38)

### 3.6.2 Location of effective load

In the above example, we computed the effective load and moment about the joint. Can we represent the effective load using just a force at a location, as in the following picture?

We know that the total force should just be the same as the sum of all the other forces, that is

\[
f_{\text{eff}} = \sum_i \begin{bmatrix} 0 \\ w_i y_i \Delta x \end{bmatrix}
\]  \hspace{1cm} (3.39)

What about the location of the force, \(L\)? We can solve for \(L\) using the fact that the effective force should generate the same moment about the joint as the distributed load. If \(f_{\text{eff}}\) is located at a distance \(L\) from the joint, then it has distance and moment vectors

\[
r_{\text{eff}} = \begin{bmatrix} L_{\text{eff}} \\ 0 \\ 0 \end{bmatrix} \hspace{1cm} \text{and} \hspace{1cm} M_{\text{eff}} = \begin{bmatrix} \hat{i} \\ 0 \\ 0 \end{bmatrix} L_{\text{eff}} - \sum_i \begin{bmatrix} \hat{i} \\ 0 \\ 0 \end{bmatrix} w_i y_i \Delta x = L_{\text{eff}} \sum_i \begin{bmatrix} 0 \\ w_i y_i \Delta x \end{bmatrix}
\]  \hspace{1cm} (3.40)
We know that \( M_{\text{eff}} = M \) so we have

\[
- L_{\text{eff}} \sum_i \begin{bmatrix} 0 \\ 0 \\ w_i y_i \Delta x \end{bmatrix} = - \sum_i \begin{bmatrix} 0 \\ 0 \\ w_i y_i x_i \Delta x \end{bmatrix}
\]  

(3.41)

Equating the \( z \) terms gives

\[
L_{\text{eff}} \sum_i w_i y_i \Delta x = \sum_i w_i y_i x_i \Delta x
\]  

(3.42)

and solving for \( L \) gives

\[
L_{\text{eff}} = \frac{\sum_i w_i y_i x_i \Delta x}{\sum_i w_i y_i \Delta x}
\]  

(3.43)

Do the units check out? They do: the top has units of \([\text{force}] [\text{distance}]\) and the bottom has units of \([\text{force}]\).

### 3.6.3 Continuous case

The previous example was an example of a load that was piecewise continuous, but we formulated it in a very specific way so as to make the connection to continuous loads. Imagine that we refine the blocks so that they become narrower and narrower, approaching a smooth curve on the top:

We will take the limit as the width of these rectangles goes to zero. What is the total force exerted by this continuous blob?

\[
f = \lim_{\Delta x \to 0} \sum_{i=0}^{L} \begin{bmatrix} 0 \\ 0 \\ -w_i y_i \Delta x \end{bmatrix}
\]

or, looking at the \( y \) component,

\[
f_y = - \lim_{\Delta x \to 0} \sum_{i=0}^{L} w_i y_i \Delta x
\]  

(3.44)

This may look familiar from calculus. In the limiting case, we have

\[
x_i \to x \quad w_i \to w(x) \quad y_i \to y(x) \quad \Delta x \to dx \quad \sum_i \to \int_0^L
\]

(3.45)

so in the limit, our total force can be evaluated as the integral

\[
f_y = - \int_0^L w(x) y(x) \, dx
\]

or, for all three components,

\[
f = - \int_0^L \begin{bmatrix} 0 \\ 0 \\ w(x) y(x) \end{bmatrix} \, dx
\]  

(3.46)

where the integral sign can safely be taken into each term in the vector. Now, let us do the same thing for the moment: taking the limit as \( \Delta x \to 0 \), we get

\[
M_z = - \lim_{\Delta x \to 0} \left[ \sum_i \begin{bmatrix} 0 \\ 0 \\ w_i y_i x_i \Delta x \end{bmatrix} \right] = - \int_0^L \begin{bmatrix} 0 \\ 0 \\ w(x) y(x) x \end{bmatrix} \, dx
\]  

(3.47)

What is the location of the effective force? Doing the exact same analysis gives us that

\[
L = \frac{\int_0^L w(x) y(x) \, dx}{\int_0^L w(x) y(x) \, dx}
\]  

(3.48)
3.6.4 Generalized case

We used the boxes as a specific case for a continuous load, but we could have loading scenarios that are more complex. Let us consider a generalized distributed load \( f(x) \) where \( f(x) \) has units of force per length. Then we have

\[
f_{\text{eff}} = \int f(x) \, dx
\]

(3.49)

(where \( f(x) \, dx \) is the differential force element) Let us take the moment about the origin; then what is our distance vectors? The differential force element is always acting at \( x \), so we just have \( r(x) = x \). Then, we can write that

\[
M_{\text{eff}} = \int x \times f(x) \, dx
\]

(3.50)

Example 3.6

As a sanity check, consider a load:

\[
\begin{align*}
\text{with } y(x) &= H & \text{and } w(x) &= w_0.
\end{align*}
\]

Then:

\[
F = \int_0^L w(x) \, y(x) \, dx = \int_0^L w_0 \, H \, dx = w_0 H L
\]

(3.51)

as we expect, and

\[
M_z = \int_0^L w(x) \, y(x) \, x \, dx = wH \int_0^L x \, dx = \frac{1}{2} w_0 H x^2 \bigg|_0^L = \frac{1}{2} w_0 H L^2
\]

(3.52)

Now, plug in to find the location of the effective force:

\[
L = \frac{M_z}{F} = \frac{w_0 H L^2 / 2}{w_0 H L} = \frac{L}{2}
\]

(3.53)

so the force is acting halfway, exactly as we expect.

Example 3.7

Consider the following load

\[
\begin{align*}
\text{Consider the following load}
\end{align*}
\]
where \( w(x) = w_0 \cos(\pi x/2L) \).

- force:
  \[
  \int_0^L w_0 \cos(\pi x/2L) \, dx = w_0 \left( \frac{2L}{\pi} \sin(\pi x/2L) \right)_0^L = \frac{2Lw_0}{\pi} \sin(\pi/2) = \frac{2Lw_0}{\pi}
  \]  
  (3.54)

- moment: use integration by parts
  \[
  \int_0^L w_0 \cos(\pi x/2L) \times dx = \int_0^L w_0 \cos(\pi x/2L) \frac{du}{\nu} \times dx
  \]
  \[
  = \left. \frac{w_0 2Lx}{\pi} \sin(\pi x/2L) \right|_0^L - \int_0^L \frac{2Lw_0}{\pi} \sin(\pi x/2L) \, dx
  \]
  \[
  = \frac{w_0 2L^2}{\pi} + \frac{4L^2 w_0}{\pi^2} \cos(\pi x/2L) \bigg|_0^L
  \]
  \[
  = \frac{w_0 2L^2}{\pi} - \frac{4L^2 w_0}{\pi^2} = \frac{-2w_0 L^2(\pi - 2)}{\pi^2}
  \]  
  (3.57)

- effective location:
  \[
  L = \frac{2w_0 L^2(\pi - 2)}{\pi^2} \times \frac{\pi}{2Lw_0} = \frac{L(\pi - 2)}{\pi}
  \]  
  (3.59)

### 4 Rigid Body Equilibrium

This section is where we start to solve more practical and interesting problems. We have built up all the machinery that we need to do static analysis on many types of objects.

#### 4.1 Equilibrium equations

With point equilibrium, we were concerned only about equilibrium of forces. Now, (as we have been alluding to) we will solve for moment equilibrium as well. The governing equations for the equilibrium of a single rigid body in 2D or 3D are

\[
\sum F = 0 \quad \sum M = 0
\]  
(4.1)

that is, the sum of the forces and moments must equal to zero. These are vector equations, so how many actual equations is this?

- 2D [2 force equations \((x,y)\)] + [1 moment equation \((z)\)] = 3 equations
- 3D [3 force equations \((x,y,z)\)] + [3 moment equations \((x,y,z)\)] = 6 equations

There are a few things to note:

1. Even for 2D problems, you should do everything in 3D. It’s more systematic and makes it easier to keep track of moments.

2. In general, you should be able to solve these equations easily, but occasionally you may need to use Cramer’s rule.

3. In general, when solving by hand, you will only have to solve “reduced” problems in 3D with fewer equations and unknowns.
4.1.1 Choosing a reference point

We have talked a lot about computing the moments of vectors “about a point.” Which point should you choose? It turns out that if

\[ \sum f = 0 \quad \sum M = 0 \]

(4.2)

where the moments computed about one point are zero, the moments computed about all points are zero. We’ll prove this:

Suppose a body is subjected to a system of forces (and moments) and is in equilibrium, where all moments are computed about the origin:

\[ \sum f_i = 0 \quad \sum M^0_i = \sum r_i \times f_i = 0 \]

(4.3)

Then the sum of the moments computed about a different point (b) will be zero as well:

\[ \sum M^b_i = \sum (r_0 + r_i) \times f_i = \sum r_0 \times f_i + \sum r_i \times f_i^0 = r_0 \times \sum f_i^0 = 0 \]

(4.4)

There are a few things to note:

1. Because you can choose any reference point, always try to choose one that makes computing moments easy.
2. If you use all of your force equations, you can only use the moment equation once.
3. Alternatively, you can use two moment equations, but then you can only use one of your force equations.

4.2 2D Free-body diagrams

Free-body diagrams are crucial for describing and quantifying the problem, especially when we have to take into account the reactions of constraints.

4.2.1 Reaction forces and moments

**Pivots**

A pivot is a constraint that restricts motion in the x and y direction, but allows rotation in the z direction. We draw pivots like this:
Because a pivot restricts motion in two directions, it has two force unknowns. It allows rotation, so it has no moment unknowns. Therefore, we can replace a pivot with a force that has an x and y component.

**Rollers**

A roller is a constraint that restricts motion normal to the surface, but allows motion tangent to the surface. It also allows rotation in the z direction. We draw rollers like this:

Because a roller restricts motion on only one direction, it has only one force unknown. Therefore, we replace a pivot with a force where the direction is known (the normal vector to the surface) but the magnitude is not. For example, on a flat surface,

**Fixed joint**

A fixed joint is a constraint that restricts motion in two directions. It also restricts rotation. We might draw a fixed joint like this:

Because motion is restricted in two directions, the force must have two unknowns. Because rotation is restricted as well, the moment has one unknown.

**Collar**

A collar is a constraint that restricts motion in one direction, and also restricts rotation. We might draw a collar like this:

What kind of reaction force would we use here? Remember, location is only constrained in one direction, so we can only have one force unknown – it will end up being the same as a roller. The difference is that rotation is also constrained, so we will also include a couple moment.

There are a wide variety of types of constraints, and it will often be up to you to determine how to replace a given constraint with an equivalent reaction force and moment. It may not always be obvious what kind of reaction is necessary. For instance, consider this crowbar applied to these smooth surfaces.
How do we replace this with reaction forces and moments? The keyword is “smooth” – we know that the surfaces can only exert forces normal to them, so we treat them as sliders.

4.2.2 Static indeterminacy and insufficient constraints:

Consider the following beam constrained with two rollers and subjected to a loading force \( f \). What are the reactions at the rollers?

\[
\begin{align*}
\mathbf{f}_{\text{rola}} &= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\
\mathbf{f}_{\text{rolb}} &= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}
\end{align*}
\]

We only have two force unknowns, which means we have two unknowns and three equations, so we cannot solve our system. In this case, it may seem pretty obvious that the system is unsolvable: with nothing holding the beam back, the horizontal component of the force will push it away. In this case, we say that the system has insufficient constraints.

Consider the following beam subjected to a loading force \( f \).

\[
\begin{align*}
\mathbf{f}_{\text{piva}} &= \begin{bmatrix} f_{\text{piva},x} \\ f_{\text{piva},y} \\ 0 \end{bmatrix} \\
\mathbf{f}_{\text{pivb}} &= \begin{bmatrix} f_{\text{pivb},x} \\ f_{\text{pivb},y} \\ 0 \end{bmatrix}
\end{align*}
\]

Can we solve the system for the reaction at the joints? No: we have four unknowns but only three equations. It does not mean that the system is unphysical, it just means that there are an infinite number of equally physical solutions. In this case, we say that the system is statically indeterminate.

Consider the following two cases:

How many unknowns do we have for these cases? In the first we have three force unknowns, in the second we have two force unknowns and one moment unknown, so both are solvable. We'll see these a lot: they are referred to as "simply supported" (left) and "cantilever" (right).

4.3 2D solution strategy

Just like with point equilibrium, we have a recipe for solving 2D equilibrium problems.

1. Identify unknowns: they may be specified in the problem, or you may need to find the reaction forces/moments of constraints in the problem.
2. Draw a free body diagram: remember to replace all of the constraints with the appropriate reaction force/moment.

3. Write governing equations:
   - Force equilibrium
   - Moment equilibrium
   - Geometric
   - Constitutive

4. Solve: easiest way possible, Cramer's rule if you must.

5. Sanity check: make sure the units come out right, and substitute a couple of "easy" values to see what happens to the solution.

6. Substitute numbers

We'll cement this solution strategy by doing a couple of examples in 2D.

### 4.4 2D examples

**Example 4.1**

Consider the following rigid body that is supported by a roller and a rigidly-attached collar sliding along a rod.

![Diagram of a rigid body supported by a roller and a collar sliding along a rod.](image)

Determine the support reactions.

Proced formally:

1. Identify unknowns: at the smooth support we have $f_{roll,y}$ The collar exerts a force along the normal direction $f_{coll}$, and a moment $M_{coll}$.

2. Draw free-body diagram:
3. Write equations: The equations of force equilibrium are:

\[
\begin{bmatrix}
\frac{f_{\text{coll}}}{\sqrt{2}} - f_{\text{roll}} \\
\frac{f_{\text{coll}}}{\sqrt{2}} - f \\
0 \\
0
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]  
(4.5)

Let us select the collar as a point for computing the moments. Our distance vectors are:

\[
r = \begin{bmatrix}
L \\
0 \\
0
\end{bmatrix} \quad r_{\text{roll}} = \begin{bmatrix}
2L \\
L \\
0
\end{bmatrix}
\]  
(4.6)

so the moments are

\[
M = r \times f = \begin{bmatrix}
0 \\
0 \\
-Lf
\end{bmatrix} \quad M_{\text{roll}} = r_{\text{roll}} \times f_{\text{roll}} = \begin{bmatrix}
0 \\
0 \\
L f_{\text{roll}}
\end{bmatrix}
\]  
(4.7)

Our equation of moment equilibrium is, then:

\[
M + M_{\text{roll}} + \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]  
(4.8)

4. We can write the equilibrium equations in matrix form as:

\[
\begin{bmatrix}
-1 & 1/\sqrt{2} & 0 \\
0 & 1/\sqrt{2} & 0 \\
L & 0 & 1
\end{bmatrix} \begin{bmatrix}
f_{\text{roll}} \\
f_{\text{coll}} \\
M_{\text{coll}}
\end{bmatrix} = \begin{bmatrix}
0 \\
f \\
Lf - M
\end{bmatrix}
\]  
(4.9)

where we can solve using Cramer’s rule.

However, this is a simple system that can be solved easily simply by substitution: From y-force equilibrium:

\[
f_{\text{coll}} = f \sqrt{2}
\]  
(4.10)

Substituting into x-force equilibrium gives:

\[
f_{\text{roll}} = \frac{f_{\text{coll}}}{\sqrt{2}} = f
\]  
(4.11)

Substituting into moment equilibrium gives:

\[
M_{\text{coll}} = L f - M - L f_{\text{roll}} = L f - M - L f_{\text{roll}} = -M
\]  
(4.12)

So we finally have

\[
\begin{bmatrix}
f_{\text{coll}} \\
f_{\text{roll}} \\
M_{\text{coll}}
\end{bmatrix} = \begin{bmatrix}
f \\
-f \\
0
\end{bmatrix} \quad \begin{bmatrix}
f_{\text{coll}} \\
f_{\text{roll}} \\
M_{\text{coll}}
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
-M
\end{bmatrix}
\]  
(4.13)

5. Sanity check:

- Units? √
Example 4.2

Find the reactions at the pivot and the roller supporting the following rigid body:

1. Unknowns: $f_{piv,x}, f_{piv,y}, f_{rol}$

2. Free body diagram:
   - first, let's reduce the distributed load: find the total force
     \[
     F_{eff,y} = \int_0^L \frac{w_0x}{L} \, dx = \frac{w_0x^2}{2L} \bigg|_0^L = \frac{w_0L}{2} \tag{4.14}
     \]
     find the total moment:
     \[
     M_{eff,x} = -\int_0^L \frac{w_0x^2}{L} \, dx = \frac{w_0x^3}{3L} \bigg|_0^L = \frac{w_0L^2}{3} \tag{4.15}
     \]
     and the effective location:
     \[
     L_{eff} = -\frac{M_{eff}}{F_{eff}} = \frac{w_0L^2}{3} \frac{2}{3L} = \frac{2L}{3} \tag{4.16}
     \]
     and use this in our FBD:
3. Write equations:

force equilibrium:
\[
\begin{bmatrix}
  f_{piv,x} \\
  f_{piv,y} + f_{rol,y} - f - w_0L/2
\end{bmatrix} = \begin{bmatrix}
  0 \\
  0
\end{bmatrix}
\]  

(4.17)

moment equilibrium: select the pivot as the point, so the distance vectors are
\[
\begin{align*}
  \mathbf{r} &= \begin{bmatrix} L/2 \\ H/2 \end{bmatrix} \\
  \mathbf{r}_{\text{eff}} &= \begin{bmatrix} 2L/3 \\ 0 \\ 0 \end{bmatrix} \\
  \mathbf{r}_{\text{rol}} &= \begin{bmatrix} L \\ 0 \\ 0 \end{bmatrix}
\end{align*}
\]  

(4.18)

and the moments are
\[
\begin{align*}
  \mathbf{M} &= \begin{bmatrix} 0 \\ 0 \\ -fL/2 \end{bmatrix} \\
  \mathbf{M}_{\text{eff}} &= \begin{bmatrix} 0 \\ 0 \\ -w_0L^2/3 \end{bmatrix} \\
  \mathbf{M}_{\text{rol}} &= \begin{bmatrix} 0 \\ 0 \\ Lf_{\text{rol},y} \end{bmatrix}
\end{align*}
\]  

(4.19)

so the moment equilibrium is
\[
\begin{bmatrix}
  0 \\
  0 \\
  -fL/2 - w_0L^2/3 + Lf_{\text{rol},y}
\end{bmatrix} = \begin{bmatrix}
  0 \\
  0 \\
  0
\end{bmatrix}
\]  

(4.20)

4. Solve: from x equilibrium we have that \( f_{piv,x} = 0 \). What does this mean?

moment equilibrium gives:
\[
\begin{align*}
  f_{\text{rol},y} &= \frac{1}{2}f + \frac{1}{3}w_0L \\
  \end{align*}
\]  

(4.21)

y force equilibrium gives:
\[
\begin{align*}
  f_{piv,y} &= f + \frac{1}{2}w_0L - f_{\text{rol},y} = f + \frac{1}{2}w_0L - \frac{1}{2}f - \frac{1}{3}w_0L = f + \frac{w_0L}{6}
\end{align*}
\]  

(4.22)

5. Sanity check: do the units check out? Yes. If I substitute \( f = w_0 = 0 \) do I get zero for all of my reactions? Yes.

We will do one more example of rigid body equilibrium in 2D before moving to 3D:

**Example 4.3**

Consider the following system where a load \( f \) with magnitude \( f \) is applied to a beam as shown:
Determine \( w \) as a function of \( L, f, \theta_1, \theta_2 \)

1. Identify unknowns: we are asked for \( w \), but we will also need to find the reactions \( f_{rol1}, f_{rol2} \)

2. Draw FBD:

\[
\begin{align*}
\mathbf{f}_{rol2} &= \begin{bmatrix} -f_{rol2} \\ 0 \\ 0 \end{bmatrix} \\
\mathbf{r}_{rol2} &= \begin{bmatrix} L \cos \theta_1 \\ L \sin \theta_1 \\ 0 \end{bmatrix} \\
\mathbf{f}_{rol1} &= \begin{bmatrix} 0 \\ f_{rol1} \\ 0 \end{bmatrix} \\
\mathbf{f} &= \begin{bmatrix} 0 \\ -f \\ 0 \end{bmatrix} \\
\mathbf{r} &= \mathbf{r}_{rol2}/2 \\
\mathbf{f}_{cab} = \mathbf{t} \mathbf{n} &= \begin{bmatrix} t \cos \theta_2 \\ t \sin \theta_2 \\ 0 \end{bmatrix}
\end{align*}
\]

3. Write equations:
   - force equilibrium for the beam:
     \[
     \sum \mathbf{f} = \mathbf{f} + \mathbf{f}_{cab} + \mathbf{f}_{rol1} + \mathbf{f}_{rol2} = \begin{bmatrix} 0 \\ -f \\ 0 \end{bmatrix} + \begin{bmatrix} t \cos \theta_2 \\ t \sin \theta_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ f_{rol1} \\ 0 \end{bmatrix} + \begin{bmatrix} -f_{rol2} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
     \]
   - force equilibrium for the box:
     \[
     t = w
     \]
moment equilibrium: choose roller 1. compute \( r, r_{\text{rol2}} \) – all other \( r \) vectors are zero.

\[
\begin{align*}
r &= \begin{bmatrix}
\frac{L}{2} \cos \theta_1 \\
\frac{L}{2} \sin \theta_1 \\
0
\end{bmatrix} \\
r_{\text{rol2}} &= \begin{bmatrix}
L \cos \theta_1 \\
L \sin \theta_1 \\
0
\end{bmatrix}
\end{align*}
\] (4.25)

now compute the moments by taking the cross product:

\[
\begin{align*}
\mathbf{M} &= \mathbf{r} \times \mathbf{f} = \begin{bmatrix}
\hat{i} \\
\hat{j} \\
\hat{k}
\end{bmatrix} \begin{bmatrix}
\frac{L}{2} \cos \theta_1 \\
\frac{L}{2} \sin \theta_1 \\
0
\end{bmatrix} \begin{bmatrix}
0 \\
0 \\
-\frac{1}{2} L f \cos \theta_1
\end{bmatrix}
\end{align*}
\] (4.26)

\[
\begin{align*}
\mathbf{M}_{\text{rol2}} &= r_{\text{rol2}} \times f_{\text{rol2}} = \begin{bmatrix}
\hat{i} \\
\hat{j} \\
\hat{k}
\end{bmatrix} \begin{bmatrix}
L \cos \theta_1 \\
L \sin \theta_1 \\
0
\end{bmatrix} \begin{bmatrix}
0 \\
0 \\
L f_{\text{rol2}} \sin \theta_1
\end{bmatrix}
\end{align*}
\] (4.27)

so moment equilibrium gives us:

\[
\sum \mathbf{M} = \mathbf{M} + \mathbf{M}_{\text{rol2}} = \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix} + \begin{bmatrix}
0 \\
0 \\
L f_{\text{rol2}} \sin \theta_1
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\] (4.28)

4. Now solve:

moment-z equilibrium gives:

\[
\frac{1}{2} L f_{\text{rol2}} \sin \theta_1 = \frac{1}{2} L f \cos \theta_1 \implies f_{\text{rol2}} = \frac{f \cos \theta_1}{2 \sin \theta_1} = \frac{f}{2 \tan \theta_1}
\] (4.29)

force-x equilibrium gives:

\[
t \cos \theta_2 = f_{\text{rol2}} \implies t = \frac{f_{\text{rol2}}}{\cos \theta_2} = \frac{f}{2 \tan \theta_1 \cos \theta_2}
\] (4.30)

which is our answer. Just for kicks, we can also get the reaction at the first roller from the force-y equilibrium equation:

\[
f_{\text{rol1}} = f - t \sin \theta_2 = f - \frac{f}{2 \tan \theta_1 \cos \theta_2} \sin \theta_1 = f \left(1 - \frac{\tan \theta_2}{2 \tan \theta_1}\right)
\] (4.31)

### 4.5 3D Free-body diagrams

We will now turn our attention to rigid body equilibrium problems in 3D. Recall that

**Pivots** Just like for the 2D case, a pivot is a constraint that restricts translation in all directions; in the 3D case it restricts translation in \( x, y, z \). On the other hand, it allows rotation about all axes. We draw pivots like this:

![Pivot Diagram](image)

A pivot restricts three degrees of freedom of the system, and as such it introduces three force unknowns. Therefore we replace the picture of a pivot with a force that has three unknowns:

\[
f = \begin{bmatrix}
f_x \\
f_y \\
f_z
\end{bmatrix}
\]

\[
M = \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]
Journal bearing

A journal bearing is similar to the collar in 2D. A smooth journal bearing restricts motion in two directions: transverse to the hole and normal to the surface. It allows translation through the bearing. Therefore it has two force degrees of freedom. A bearing allows axial rotation, but does not allow rotation in any other direction. We will (attempt to) draw a journal bearing like this:

\[
\mathbf{M} = \begin{bmatrix} M_x \\ 0 \\ M_z \end{bmatrix}
\]
\[
\mathbf{f} = \begin{bmatrix} f_x \\ 0 \\ f_z \end{bmatrix}
\]

The bearing must include two force unknowns and two moment unknowns, so we have four unknowns altogether. (Sometimes we'll add the constraint that the bearing is not allowed to sustain any moments at all – we sometimes say that this means it is “well-designed.” In that case, we assume that the moments are zero, and so we only have two unknowns.)

Square journal bearing

The square journal bearing is similar to the regular journal bearing, except that rotation is not allowed in the axial direction either. We might draw a square journal bearing like this:

\[
\mathbf{M} = \begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix}
\]
\[
\mathbf{f} = \begin{bmatrix} f_x \\ 0 \\ f_z \end{bmatrix}
\]

No rotation is allowed at all, so the moment has three unknowns. The force vector is the same as for the journal bearing, resulting in five total unknowns.

Hinge

A hinge is similar to a journal bearing, except that it constrains motion in all directions, allowing rotation in exactly one direction. We might draw a hinge like this:

\[
\mathbf{M} = \begin{bmatrix} M_x \\ 0 \\ M_z \end{bmatrix}
\]
\[
\mathbf{f} = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix}
\]

A hinge introduces three force unknowns and two moment unknowns.

Fixed joint

A fixed joint is a joint that does not allow for any rotations or any moments. It’s kind of tricky to draw a “general” fixed joint but an example of one might look like this:

\[
\mathbf{M} = \begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix}
\]
\[
\mathbf{f} = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix}
\]
A fixed contact introduces three force and three moment unknowns, and causes the problem to be fully constrained. (If you ever encounter a problem with more than one constraint, and one of them is a fixed joint, you know right off that it will be statically indeterminant.)

There are multiple other types of 3D constraints, and it is good to get some practice figuring out how to turn a constraint into a set of reaction forces and moments.

4.6 3D solution strategy

The 3D solution strategy is identical to that for 2D; especially if you use the 3D vector formulation in 2D, you'll see that the procedure is identical. The only difference is that both force and moment balance is significant for all three dimensions, so you end up with a total of six equilibrium equations.
4.7 3D examples

Example 4.4

Consider the following frame that is mounted using a pivot, a journal bearing, and a roller and is subjected to a force \( f \) with magnitude \( f \).

![Image of a frame mounted with a pivot, journal bearing, and roller]

Determine the reactions at the supports.

I’m going to switch the order of (1) and (2) here and draw the FBD first:

1. Free body diagram:

   \[
   \begin{bmatrix}
   f_{\text{piv}} \\
   f_{\text{jou}} \\
   f_{\text{rol}}
   \end{bmatrix}
   \]

   \[
   \begin{bmatrix}
   f_{\text{pivx}} \\
   f_{\text{pivy}} \\
   f_{\text{pivz}} \\
   f_{\text{joux}} \\
   f_{\text{jouy}} \\
   f_{\text{jouz}} \\
   f_{\text{rolx}} \\
   f_{\text{rolz}}
   \end{bmatrix}
   \]

   \[
   f = \begin{bmatrix}
   0 \\
   0 \\
   -f
   \end{bmatrix}
   \]

   \[
   M_{\text{joux}} = \begin{bmatrix}
   M_{\text{joux}} \\
   0 \\
   M_{\text{jouz}}
   \end{bmatrix}
   \]

   \[
   L
   \]

2. Identify unknowns: \( f_{\text{pivx}}, f_{\text{pivy}}, f_{\text{pivz}}, f_{\text{joux}}, f_{\text{jouy}}, f_{\text{rolz}}, M_{\text{joux}}, M_{\text{jouz}} = 8 \) unknowns. Uh oh, we have too many unknowns, which means that this structure is statically indeterminate. This makes sense: imagine that the bearing was “pre-stressed”, causing an additional moment in the system.

To solve this, we need to make some assumptions about the problem. We will make the intuitive assumption that the moment exerted on the journal bearing is zero. If this assumption is safe, then we will be able to solve the problem. If not, we’ll have to go back and check our assumptions.

Assume: \( M_{\text{joux}} = M_{\text{jouy}} = 0 \). (always state assumptions clearly in your work!)

3. Write equations:

   force equilibrium:

   \[
   \begin{bmatrix}
   f_{\text{piv}} + f_{\text{jou}} + f_{\text{rol}} + f
   \end{bmatrix}
   =
   \begin{bmatrix}
   f_{\text{pivx}} + f_{\text{joux}} \\
   f_{\text{pivy}} + f_{\text{jouy}} \\
   f_{\text{pivz}} + f_{\text{jouz}} + f_{\text{rolz}} - f
   \end{bmatrix}
   =
   \begin{bmatrix}
   0 \\
   0 \\
   0
   \end{bmatrix}
   \]

   (4.32)
moment equilibrium: as usual, we must choose a reference point. Choosing the origin, 

\[
\mathbf{r} = \begin{bmatrix} -\ell \\ W \\ 0 \end{bmatrix}, \quad \mathbf{r}_{\text{jou}} = \begin{bmatrix} 0 \\ 2W \\ 0 \end{bmatrix}, \quad \mathbf{r}_{\text{rol}} = \begin{bmatrix} -L \\ 3W \\ 0 \end{bmatrix}
\]

(4.33)

so the moments are

\[
\mathbf{M} = \mathbf{r} \times \mathbf{f} = \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix} \begin{bmatrix} -\ell \\ W \\ 0 \end{bmatrix} = \begin{bmatrix} -Wf \\ -\ell f \end{bmatrix}
\]

(4.34)

\[
\mathbf{M}_{\text{jou}} = \mathbf{r}_{\text{jou}} \times \mathbf{f}_{\text{jou}} = \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix} \begin{bmatrix} 0 \\ 2W \\ 0 \end{bmatrix} = \begin{bmatrix} 2Wf_{\text{jouz}} \\ 0 \\ -2Wf_{\text{joux}} \end{bmatrix}
\]

(4.35)

\[
\mathbf{M}_{\text{rol}} = \mathbf{r}_{\text{rol}} \times \mathbf{f}_{\text{rol}} = \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix} \begin{bmatrix} -L \\ 3W \\ 0 \end{bmatrix} = \begin{bmatrix} 3Wf_{\text{rolz}} \\ Lf_{\text{rolz}} \\ 0 \end{bmatrix}
\]

(4.36)

sum of the moments:

\[
\sum \mathbf{M} = \begin{bmatrix} -Wf \\ -\ell f \end{bmatrix} + \begin{bmatrix} 2Wf_{\text{jouz}} \\ 0 \\ -2Wf_{\text{joux}} \end{bmatrix} + \begin{bmatrix} 3Wf_{\text{rolz}} \\ Lf_{\text{rolz}} \\ 0 \end{bmatrix} = \begin{bmatrix} -Wf + 2Wf_{\text{jouz}} + 3Wf_{\text{rolz}} \\ -\ell f + Lf_{\text{rolz}} \\ -2Wf_{\text{joux}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\]

(4.37)

4. Solve:

Y-force: \( f_{\text{pivy}} = 0 \)

Z-moment: \( f_{\text{joux}} = 0 \)

Y-moment: \( -\ell f + Lf_{\text{rolz}} = 0 \) \( \implies f_{\text{rolz}} = \frac{\ell f}{L} \)

X-moment: \( -Wf + 2Wf_{\text{jouz}} + 3Wf_{\text{rolz}} = 0 \) \( \implies \)

\[
f_{\text{jouz}} = \frac{1}{2}(f - 3f_{\text{rolz}}) = \frac{f}{2} - \frac{3\ell f}{2L}
\]

(4.38)

X-force: \( f_{\text{pivx}} + f_{\text{joux}} = 0 \) \( \implies f_{\text{pivx}} = 0 \)

Z-force: \( f_{\text{pivz}} + f_{\text{jouz}} + f_{\text{rolz}} = 0 \) \( \implies \)

\[
f_{\text{pivz}} = f - f_{\text{jouz}} - f_{\text{rolz}} = f - \frac{1}{2}f(1 - 3\ell/L) - \ell f/L
\]

(4.39)

\[
= f - \frac{1}{2}f(1 - 3\ell/L) - \ell f/L
\]

(4.40)

\[
= \frac{f}{2} + \frac{f\ell}{2L}
\]

(4.41)

\[
= \frac{f}{2} + \frac{f\ell}{2L}
\]

(4.42)

\textbf{Example 4.5}

Determine the tensions in the cables and the reaction of the strut for the following system.
1. Determine unknowns: \( f_{pivx}, f_{pivy}, f_{pivz}, t_{cab1}, t_{cab2}, t_{cab3} \)

2. Draw free body diagram:

where the unit vector \( n_{cab2} \) is

\[
n_{cab2} = \frac{1}{\sqrt{4W^2 + H^2}} \begin{bmatrix} 0 \\ 0 \\ -2W/H \end{bmatrix} \Rightarrow f_{cab2} = \begin{bmatrix} 0 \\ -2Wt_{cab2}/\ell \\ Ht_{cab2}/\ell \end{bmatrix}
\]  

(4.43)

and \( \ell = \sqrt{4W^2 + H^2} \).

3. Write giverning equations:

force equilibrium:

\[
\sum f = f_{piv} + w + f_{cab1} + f_{cab2} + f_{cab3} = \begin{bmatrix} f_{pivx} \\ f_{pivy} \\ f_{pivz} - w + t_{cab1} + Ht_{cab2}/\ell + f_{cab3} \end{bmatrix}
\]  

(4.44)
moment about the pivot: compute distance vectors

\[
\begin{align*}
\mathbf{r}_w &= \begin{bmatrix} 2L \\ W \\ 0 \end{bmatrix} \\
\mathbf{r}_{cab1} &= \begin{bmatrix} 2L \\ W \\ 0 \end{bmatrix} \\
\mathbf{r}_{cab2} &= \begin{bmatrix} 2L \\ 2W \\ 0 \end{bmatrix} \\
\mathbf{r}_{cab3} &= \begin{bmatrix} L \\ 2W \\ 0 \end{bmatrix}
\end{align*}
\]  

(4.45)

so the moments are:

\[
\begin{align*}
\mathbf{M}_w &= \mathbf{r}_w \times \mathbf{w} = \begin{bmatrix} \hat{i} \\ \frac{2L}{W} \\ \frac{W}{0} \end{bmatrix} = \begin{bmatrix} -wW \\ 2Lw \\ 0 \end{bmatrix} \\
\mathbf{M}_{cab1} &= \mathbf{r}_{cab1} \times \mathbf{f}_{cab1} = \begin{bmatrix} \hat{i} \\ \frac{2L}{0} \\ \frac{W}{0} \end{bmatrix} = \begin{bmatrix} 0 \\ -2Lt_{cab1} \\ 0 \end{bmatrix} \\
\mathbf{M}_{cab2} &= \mathbf{r}_{cab2} \times \mathbf{f}_{cab2} = \begin{bmatrix} \hat{i} \\ \frac{2L}{2W} \\ \frac{0}{0} \end{bmatrix} = \begin{bmatrix} 2Wt_{cab2}/\ell \\ -2Lt_{cab2}/\ell \\ -4Wt_{cab2}/\ell \end{bmatrix} \\
\mathbf{M}_{cab3} &= \mathbf{r}_{cab3} \times \mathbf{f}_{cab3} = \begin{bmatrix} \hat{i} \\ \frac{L}{0} \\ \frac{2W}{0} \end{bmatrix} = \begin{bmatrix} 2Wt_{cab3} \\ -Lt_{cab3} \\ 0 \end{bmatrix}
\end{align*}
\]  

(4.46)

(4.47)

(4.48)

(4.49)

and the moment equation is:

\[
\begin{align*}
\sum \mathbf{M} &= \begin{bmatrix} -wW + 2Wt_{cab2}/\ell + 2Wt_{cab3} \\ 2Lw - 2Lt_{cab1} - 2Lt_{cab2}/\ell - Lt_{cab3} \\ -4Wt_{cab2}/\ell \\ -4Wt_{cab2}/\ell \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\end{align*}
\]  

(4.50)

we notice right away that \( t_{cab2} = 0 \) which greatly simplifies the analysis. Rewriting the above:

\[
\begin{align*}
\sum \mathbf{M} &= \begin{bmatrix} -wW + 2Wt_{cab3} \\ 2Lw - 2Lt_{cab1} - Lt_{cab3} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\end{align*}
\]  

(4.51)

4. Solve:

moment-x:

\[-wW + 2Wt_{cab3} = 0 \implies 2t_{cab3} = \frac{w}{2}
\]  

(4.52)

moment-y:

\[2Lw - 2Lt_{cab1} - Lt_{cab3} = 0 \implies t_{cab1} = \frac{3w}{4}
\]  

(4.53)

force-x:

\[f_{pivx} = 0
\]  

(4.54)

force-y:

\[f_{pivy} - 2Wt_{cab2}/\ell = 0 \implies f_{pivy} = 0
\]  

(4.55)

force-z:

\[f_{pivz} - w + t_{cab1} + t_{cab3} = 0 \implies f_{pivz} = w - t_{cab1} - t_{cab3} = w - \frac{w}{2} - \frac{3w}{4} = -\frac{w}{4}
\]  

(4.56)

5. Sanity check: units, signs, etc. make sense.
4.8 Multi-body free-body diagrams

Up until now we have treated single rigid bodies, but now we will consider bodies that can be treated as two (or more) rigid bodies attached with some kind of connection mechanism. When we draw our free body diagrams, we must always draw at least as many diagrams as we have rigid bodies. In general, we will always draw one free body diagram for every rigid body in the problem. How do we handle the connections between them? To properly do this, we need to account for connecting forces. Some examples follow:

**Pinned Joints** Consider the following two members that are connected using a pin. Consider just the member on the right: from “its perspective” the pinned joint acts exactly like a pivot, and similarly for the member on the left. Just like with pivot constraints, we replace this with a force. The difference is that the force exerted by the left member on the right member must be equal and opposite to the force exerted by the right member on the left. So, when we draw our FBD, we split it up in the following way:

\[
\begin{align*}
\mathbf{f}_{\text{pin}} &= \begin{bmatrix} f_{\text{pinx}} \\ f_{\text{piny}} \\ 0 \end{bmatrix} \\
-\mathbf{f}_{\text{pin}} &= \begin{bmatrix} -f_{\text{pinx}} \\ -f_{\text{piny}} \\ 0 \end{bmatrix}
\end{align*}
\]

Note that each member fixes the location of the other, implying that a force is sustained by the pinned joint. On the other hand, they are free to rotate, so neither beam can exert a moment on the other.

What about when there are more than one object joined with one pin, as in the following example:

Here it may be helpful to think of the pin as an additional rigid body, where the forces acting on it are \(-f_{\text{pina}}, -f_{\text{pinb}}, -f_{\text{pinc}}\). We know by force equilibrium that we must then have

\[
f_{\text{pina}} + f_{\text{pinb}} + f_{\text{pinc}} = 0
\]

Note that this adds an additional equation. Note also that if \(f_{\text{pinc}}\) were absent, we would recover \(f_{\text{pina}} = -f_{\text{pinb}},\) exactly as we expect.

**Slider** The slider is analogous to the familiar roller joint: two members are connected with a pin, allowing motion of one member along the slot. The free body diagram will be similar to that for the roller, noting that the forces again must be equal and opposite.
We note that the fixed relative position normal to the slot requires a sustained force normal to the slot. Free relative position parallel to the slot implies that there is no force parallel to the slot. Free relative rotation implies that there is no sustained moment. So a slider introduces one force unknown and no moment unknowns.

**Frictionless contact**

Consider two members that have smooth contact with each other and a wide contact area. Keywords: “smooth” implies that there is free tangential motion, and “wide contact area” implies that rotation is restricted.

As with the slider, the force is strictly normal, which implies that there is only one force unknown. (Note – we might also require here that the force must be compressive, as the contact clearly could not contain a tensile load.) The constrained rotation implies that there is a sustained moment, giving one moment unknown. (Remember that the moment must be equal and opposite as well.)

**Fixed**

Recall our discussion of “fixed joints” from the previous section.

We recall that fixed contact implies the presence of two force unknowns and one moment unknown in 2D, and three force and three moment unknowns in 3D. Consider the following rigid body. We can think of this body as being split in half, with a “fixed joint” in between them.
There are three things worth noting:

1. Two bodies joined with a fixed joint are identical to a rigid body.
2. A fixed joint adds the maximum number of unknowns, and splitting a body into two adds the maximum number of unknowns.
3. You can always find the internal connecting force/moment in a rigid body by splitting it in two pieces and solving for the connecting forces.

We will make considerable use of this kind of technique in subsequent sections.

4.9 Multi-body equilibrium

Previously, we always had just one 2D/3D moment and one 2D/3D force equilibrium equation. We will always need at least as many force/moment equilibrium eqs as we have rigid bodies, but we can be clever about how we choose them. For instance, consider the following multi-body structure:

![Multi-body structure diagram]

Here, we can do equilibrium for the two beams separately, or we can do equilibrium for the entire combined structure and one of the members. It generally doesn't matter, and you can be clever about how you set your problems up.

4.10 Multi-body solution strategy

We will follow a methodical procedure for solving these problems, similarly to the procedure used for problems solved in previous sections.

1. Determine unknowns: (requested quantities, reaction forces, connecting forces)
2. Draw FBD for each rigid body
3. Write equations:
   - Force equilibrium for each member (and one for each multi-pinned joint, if necessary)
   - Moment equilibrium for each member
   - Other: Constitutive, geometric, etc.
4. Solve
5. Sanity check
6. Substitute values
4.11 Multi-body examples

Example 4.6

Consider the following frame:

Find all of the reaction and connection forces:

1. Unknowns: \( f_{fixx}, f_{fixy}, M_{fixz}, f_{pinx}, f_{piny}, f_{roly} \)

2. FBD

\[
\begin{align*}
\mathbf{f}_{fix} &= \begin{bmatrix} f_{fixx} \\ f_{fixy} \\ 0 \end{bmatrix} \\
\mathbf{M}_{fix} &= \begin{bmatrix} 0 & 0 & -L_1 f_1 \cos \theta \\ 0 & M_{fixz} & 0 \\ -L_1 f_1 \cos \theta & 0 & 0 \end{bmatrix} \\
\mathbf{f}_{pinx} &= \begin{bmatrix} f_{pinx} \\ f_{piny} \\ 0 \end{bmatrix} \\
\mathbf{f}_{roly} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
f_1 &= \begin{bmatrix} f_1 \sin \theta \\ -f_1 \cos \theta \end{bmatrix} \\
f_2 &= \begin{bmatrix} 0 \\ -f_2 \end{bmatrix}
\end{align*}
\] (4.58)

3. Equations:

forces:

\[
\begin{align*}
\sum \mathbf{f} &= \begin{bmatrix} f_{fix} + f_{pinx} + f \sin \theta \\ f_{fix} + f_{piny} - f_1 \cos \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\sum \mathbf{f} &= \begin{bmatrix} -f_{piny} - f_2 + f_{roly} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{align*}
\] (4.59)

moments: take about the fixed joint and then about the pin:

distance vectors:

\[
\begin{align*}
\mathbf{r}_1 &= \begin{bmatrix} f_1 \\ 0 \\ 0 \end{bmatrix} \\
\mathbf{r}_{pin} &= \begin{bmatrix} L_1 \\ 0 \\ 0 \end{bmatrix} \\
\mathbf{r}_2 &= \begin{bmatrix} 0 \\ \ell_2 \\ 0 \end{bmatrix} \\
\mathbf{r}_{roly} &= \begin{bmatrix} 0 \\ 0 \\ L_2 f_{roly} \end{bmatrix}
\end{align*}
\] (4.60)

moments:

\[
\begin{align*}
\mathbf{M}_1 &= \begin{bmatrix} 0 & 0 & -L_1 f_1 \cos \theta \\ 0 & 0 & 0 \end{bmatrix} \\
\mathbf{M}_{pin} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\mathbf{M}_2 &= \begin{bmatrix} 0 & 0 & -\ell_2 f_2 \\ 0 & 0 & 0 \end{bmatrix} \\
\mathbf{M}_{roly} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{align*}
\] (4.61)
summing the moments for beam 1:

\[ \sum M = M_1 + M_{\text{pin}} + M_{\text{fix}} = \begin{bmatrix} 0 \\ 0 \\ -\ell_1 f_1 \cos \theta + L_1 f_{\text{piny}} + M_{\text{fix}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \] (4.62)

for beam 2:

\[ \sum M = M_2 + M_{\text{rol}} = \begin{bmatrix} 0 \\ 0 \\ -\ell_2 f_2 + L_2 f_{\text{rol}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \] (4.63)

4. Solve:

beam 2 moment z:

\[ f_{\text{rol}} = \frac{f_2}{L_2} \] (4.64)

beam 2 force x:

\[ f_{\text{pinx}} = 0 \] (4.65)

beam 2 force y:

\[-f_{\text{piny}} - f_2 + f_{\text{rol}} = 0 \implies f_{\text{piny}} = f_2 \left( \frac{L_2}{L_2} - 1 \right) \] (4.66)

beam 1 moment z:

\[-\ell_1 f_1 \cos \theta + L_1 f_{\text{piny}} + M_{\text{fix}} = 0 \implies M_{\text{fix}} = \ell_1 f_1 \cos \theta - L_1 f_2 \left( \frac{L_2}{L_2} - 1 \right) \] (4.67)

beam 1 force x:

\[ f_{\text{fixx}} + f_{\text{pinx}} + f_1 \sin \theta = 0 \implies f_{\text{fixx}} = -f_1 \sin \theta \] (4.68)

beam 1 force y:

\[ f_{\text{fixy}} + f_{\text{piny}} - f_1 \cos \theta = 0 \implies f_{\text{fixy}} = f_1 \cos \theta - f_2 \left( \frac{L_2}{L_2} - 1 \right) \] (4.69)

**Example 4.7**

Consider the following frame connected using small pulleys and pinned joints. Find all connecting and reaction forces:
1. Identify unknowns: \( f^a_x, f^a_y, f^b_x, f^b_y, f^c_x, f^c_y, f^c_{b1x}, f^c_{b2x}, f^c_{b1y}, f^c_{b2y}, f^e_{c1x}, f^e_{c1y}, f^e_{c2x}, f^e_{c2y} \) – 12 unknowns.

2. Free body diagram:
   pulleys:
   \[
   \begin{bmatrix}
   0 \\
   -w \\
   0
   \end{bmatrix}
   \begin{bmatrix}
   w \\
   0 \\
   0
   \end{bmatrix}
   \begin{bmatrix}
   0 \\
   w \\
   0
   \end{bmatrix}
   \begin{bmatrix}
   0 \\
   0 \\
   -w
   \end{bmatrix}
   \]
   beam 1:
   \[
   f^a = \begin{bmatrix}
   f^a_x \\
   f^a_y \\
   0
   \end{bmatrix}
   \]
   \[
   -f^b = \begin{bmatrix}
   -f^b_x \\
   -f^b_y \\
   0
   \end{bmatrix}
   \]
   \[
   f^c_{b1} = \begin{bmatrix}
   f^c_{b1x} \\
   f^c_{b2x} \\
   0
   \end{bmatrix}
   \]
   beam 2:
   \[
   f^d = \begin{bmatrix}
   0 \\
   w \\
   0
   \end{bmatrix}
   \]
   \[
   f^c_{b2} = \begin{bmatrix}
   f^c_{b2x} \\
   f^c_{b2y} \\
   0
   \end{bmatrix}
   \]

3. Write equations:
force equilibrium – pulleys:

\[ \sum \mathbf{f} = \left[ \begin{array}{c} f_x^b + \mathbf{w} \\ f_y^b - \mathbf{w} \\ \end{array} \right] = \mathbf{0} \quad \sum \mathbf{f} = \left[ \begin{array}{c} -f_{\text{pulley}}^c - \mathbf{w} \\ f_{\text{pulley}}^c - \mathbf{w} \\ \end{array} \right] = \mathbf{0} \quad (4.70) \]

force equilibrium – beams:

\[ \sum \mathbf{f}_{\text{b}1} = \left[ \begin{array}{c} f_{\text{a}x} \\ f_{\text{c}x} - f_{\text{b}x} \\ f_{\text{c}b_{1x}} \\ f_{\text{c}b_{1y}} - f_{\text{b}y} \\ \end{array} \right] = \mathbf{0} \quad \sum \mathbf{f}_{\text{b}2} = \left[ \begin{array}{c} f_{\text{c}x} + f_{\text{b}x} \\ f_{\text{c}x} - \mathbf{w} + f_{\text{b}x} \\ f_{\text{c}b_{2y}} \\ \end{array} \right] = \mathbf{0} \quad (4.71) \]

force equilibrium – pinned joint:

\[ \sum \mathbf{f} = \mathbf{f}_{\text{c}b_{1}} + \mathbf{f}_{\text{c}b_{2}} + \mathbf{f}_{\text{pulley}} = \left[ \begin{array}{c} f_{\text{c}b_{1x}} + f_{\text{c}b_{2x}} + f_{\text{pulley}x} \\ f_{\text{c}b_{1y}} + f_{\text{c}b_{2y}} + f_{\text{pulley}y} \\ \end{array} \right] = \mathbf{0} \quad (4.72) \]

moment equilibrium – pulleys:

\[ \Rightarrow \quad \text{moment is trivially zero because tensions are equal.} \]

moment equilibrium – beams:

\[ \Rightarrow \quad \text{choose pivots for both, and compute distance vectors:} \]

\[ \mathbf{r}^b = \left[ \begin{array}{c} L \\ 0 \\ 0 \\ \end{array} \right] \quad \mathbf{r}^c_{\text{b}1} = \left[ \begin{array}{c} 2L \\ 0 \\ 0 \\ \end{array} \right] \quad \mathbf{r}^c_{\text{b}2} = \left[ \begin{array}{c} L \\ H/2 \\ 0 \\ \end{array} \right] \quad \mathbf{r}^c_{\text{b}2} = \left[ \begin{array}{c} 2L \\ H \\ 0 \\ \end{array} \right] \quad (4.73) \]

so the moments are:

\[ \mathbf{M}^b = \left[ \begin{array}{c} \mathbf{i} \\ L \\ \end{array} \right] \quad \mathbf{M}_{\text{b}1} = \left[ \begin{array}{c} \mathbf{i} \\ 2L \\ f_{\text{c}b_{1x}} \\ f_{\text{c}b_{1y}} \\ \end{array} \right] \quad \mathbf{M}_{\text{b}2} = \left[ \begin{array}{c} \mathbf{i} \\ 2L \\ f_{\text{c}b_{2x}} \\ f_{\text{c}b_{2y}} \\ \end{array} \right] \quad (4.74) \]

\[ \mathbf{M}^d = \left[ \begin{array}{c} \mathbf{i} \\ L \end{array} \right] \quad \mathbf{M}_{\text{b}1} = \left[ \begin{array}{c} \mathbf{i} \\ 2L \\ f_{\text{c}b_{1x}} \\ f_{\text{c}b_{1y}} \\ \end{array} \right] \quad \mathbf{M}_{\text{b}2} = \left[ \begin{array}{c} \mathbf{i} \\ 2L \\ f_{\text{c}b_{2x}} \\ f_{\text{c}b_{2y}} \\ \end{array} \right] \quad (4.75) \]

giving the equilibrium equations

\[ \sum \mathbf{M}_{\text{b}1} = \left[ \begin{array}{c} 0 \\ 0 \\ 0 \\ \end{array} \right] \quad \sum \mathbf{M}_{\text{b}2} = \left[ \begin{array}{c} 0 \\ 0 \\ 0 \\ \end{array} \right] \quad (4.76) \]

4. Solve:

pulley force equilibrium:

\[ f_{x}^b = -\mathbf{w} \quad f_{y}^b = \mathbf{w} \quad f_{\text{pulley}x} = \mathbf{w} \quad f_{\text{pulley}y} = \mathbf{w} \quad (4.77) \]

beam 1 moment z:

\[ 2L f_{b1y}^c - L \mathbf{w} = 0 \quad \Rightarrow \quad f_{b1y}^c = \frac{w}{2} \quad (4.78) \]
pinned joint force y:

\[(w/2) + f_{b2y}^c + (w) = 0 \implies f_{b2y}^c = -\frac{3w}{2} \]  \hspace{1cm} (4.79)

beam 2 moment z:

\[Lw + 2L(-3w/2) - Hf_{b2x}^c = 0 \implies f_{b2x}^c = \frac{2Lw}{H} \]  \hspace{1cm} (4.80)

pinned joint force x:

\[
f_{b1x}^c + \frac{f_{b2x}^c}{-2Lw/H} + \frac{f_{pullx}}{w} = 0 \implies f_{b1x}^c = w\left(\frac{2L}{H} - 1\right) \]  \hspace{1cm} (4.81)

beam 1 force x:

\[
f_x^a - f_x^b + \frac{f_{b1x}^c}{-w} + \frac{f_{b2x}^c}{w(2L/H - 1)} = 0 \implies f_x^a = -\frac{w2L}{H} \]  \hspace{1cm} (4.82)

beam 1 force y:

\[
f_y^a - f_y^b + \frac{f_{b2y}^c}{-w} - \frac{3w}{2} = 0 \implies f_y^a = \frac{w}{2} \]  \hspace{1cm} (4.83)

beam 2 force x and y:

\[
f_x^e + \frac{f_{b2x}^c}{-2Lw/H} = 0 \implies f_x^e = \frac{2Lw}{H} \]

\[
f_y^e + \frac{f_{b2y}^c}{-3w/2} = 0 \implies f_y^e = \frac{w}{2} \]  \hspace{1cm} (4.84)
5 Trusses

Now that we have introduced rigid body equilibrium with multiple connected bodies, we can extend these methods to talk about more complex and realistic structures. One type of structure that is very useful and readily amenable to static force analysis is a truss structure. We introduce trusses with the following definition:

**Definition 5.1.** A truss is a structure that is made up entirely of two-force members joined with smooth pins.

Recall that a two-force member is a member for which the load is always parallel to its direction. Here is an example of a simple truss:

We make the following notes:

1. All loads applied at the nodes of the structure. When working with trusses, we do not allow loads to be applied in the middle of a section; then the beam is no longer a two force member. When working with distributed loads, we divide the distributed load into point loads acting at the joints.

2. Because the members are, by definition, two force members, we know that each member is automatically in equilibrium. Therefore we don't need to do rigid body equilibrium individually.

3. The equilibrium equations that we will solve are either at the joints or at the supports.

There are two methods of classical truss analysis: the method of joints and the method of sections.

5.1 Method of joints

The method of joints is nothing more than simply solving force equilibrium at each point. It is frequently a little tedious—especially for complicated trusses—but it will give you everything you need to know about a system. In general the method of joints is what you want to use if you need to know the tensions in every member of the system. For instance, if you are given a truss and you need to figure out which member will carry the greatest load, you'll probably want to use the method of joints.

5.1.1 Solution Strategy

As usual, we'll follow a recipe to solve the problem.

1. Determine unknowns: tensions in members, reactions at constraints.

2. Draw free body diagram:
For the entire truss as one rigid body
For each pinned joint

(3) Write equations
- Force+moment equilibrium for the entire truss as one rigid body
- Force equilibrium for each joint

(4) Solve the system

(5) Sanity check: units (as usual). Also check the sign of the tensions in the members, remembering that negative tension is equivalent to compression. Ask whether it makes sense.

(6) Substitute numbers

5.1.2 Examples

Example 5.1

Consider the following truss:

Determine the tensions in the members and the reactions at the pivots.

(1) Unknowns: \( t_1, t_2, t_3, t_4, t_5, t_6, f_{\text{pivx}}^a, f_{\text{pivy}}^a, f_{\text{pivx}}^c, f_{\text{pivy}}^c \) = 10 unknowns

(2) Free body diagram:
(3) Equations: force equilibrium about a,b,c,d,e = [5 joints] × [2 equations per joint] = 10 equations

node a:
\[
\begin{bmatrix}
 f_{a}^{x} + t_1 \cos \theta \\
 f_{a}^{y} - t_1 \sin \theta
\end{bmatrix} = 0
\]  
(5.1)

node b:
\[
\begin{bmatrix}
 -t_1 \cos \theta - t_2 \cos \theta + t_4 \cos \theta \\
 t_3 \sin \theta - t_2 \sin \theta - t_3 - t_4 \sin \theta
\end{bmatrix} = 0
\]  
(5.2)

node c:
\[
\begin{bmatrix}
 f_{c}^{x} + t_2 \cos \theta + t_5 \\
 f_{c}^{y} + t_2 \sin \theta
\end{bmatrix} = 0
\]  
(5.3)

node d:
\[
\begin{bmatrix}
 -t_5 + t_6 \\
 t_3
\end{bmatrix} = 0
\]  
(5.4)

node e:
\[
\begin{bmatrix}
 -t_4 \cos \theta - t_6 \\
 t_4 \sin \theta - w
\end{bmatrix} = 0
\]  
(5.5)

(4) Solve:

e-y:
\[
t_4 = \frac{w}{\sin \theta}
\]  
(5.6)

e-x:
\[
t_6 = -t_4 \cos \theta = -\frac{w \cos \theta}{\sin \theta} = -\frac{w}{\tan \theta}
\]  
(negative, so it’s in compression)  
(5.7)

d-x:
\[
t_5 = t_6 = -\frac{w}{\tan \theta}
\]  
(5.8)

d-y:
\[
t_3 = 0
\]  
(ah ha! there’s our zero-force member)  
(5.9)

rats! we’re out of easy equations to solve. But, since we know \( t_3 \) and \( t_4 \), we can solve at node b fairly easily:

b:
\[
\begin{bmatrix}
 t_1 \cos \theta + t_2 \cos \theta \\
 t_1 \sin \theta - t_2 \sin \theta
\end{bmatrix} = \begin{bmatrix}
 t_2 \cos \theta \\
 t_2 \sin \theta
\end{bmatrix} = \begin{bmatrix}
 t_4 / \tan \theta \\
 w
\end{bmatrix}
\]  
(5.10)

writing in matrix form:
\[
\begin{bmatrix}
 \cos \theta & \cos \theta \\
 \sin \theta & -\sin \theta
\end{bmatrix} \begin{bmatrix}
 t_1 \\
 t_2
\end{bmatrix} = \begin{bmatrix}
 t_4 / \tan \theta \\
 w
\end{bmatrix}
\]  
(5.11)

check the determinant? \(- \cos \theta \sin \theta - \cos \theta \sin \theta = -2 \cos \theta \sin \theta \) ok as long as \( 0 \leq \theta \leq 90^\circ \). now we can use our old friend Cramer’s rule to solve:

\[
t_1 = -\frac{1}{2 \cos \theta \sin \theta} \det \begin{bmatrix}
 w / \tan \theta & \cos \theta \\
 w & -\sin \theta
\end{bmatrix} = -\frac{-w \cos \theta - w \cos \theta}{2 \cos \theta \sin \theta} = \frac{t_1}{\sin \theta} = \frac{w}{\sin \theta}
\]  
(5.12)

\[
t_2 = -\frac{1}{2 \cos \theta \sin \theta} \det \begin{bmatrix}
 \cos \theta & w / \tan \theta \\
 \sin \theta & w
\end{bmatrix} = -\frac{-w \cos \theta - w \cos \theta}{2 \cos \theta \sin \theta} = \frac{t_2}{\sin \theta} = 0
\]  
(5.13)
hey look, another zero force member!

now it's easy to get the reactions, we just have

\[
\begin{align*}
    f_\text{pixx}^c &= -t_5 = -\frac{w}{\tan \theta} \\
    f_\text{pixy}^c &= 0 \\
    f_\text{pixx}^a &= -t_1 \cos \theta = \frac{w}{\tan \theta} \\
    f_\text{pixy}^b &= t_1 \sin \theta = w
\end{align*}
\]  

(5.14)

(5) Sanity check: does it make sense that beams 2 and 3 would be zero-force members? Yes and no, because if we removed them the structure would buckle. This means that if these beams are carrying a load it is just applying a stabilizing force.

### 5.2 Loading-free members

An important aspect of truss analysis is the ability to spot members that do not or cannot carry any load. These are frequently called “zero-force” members, although we will distinguish between zero-force members and stabilizing members.

#### 5.2.1 Zero-force members

Consider the following truss (incidentally taken from one of the gables of the Grand Lodge resort at Breck).

Upon inspection, we notice that this truss has some serious problems.

1. The lower two sub-trusses are clearly free to rotate. Even though this truss might be in equilibrium, it will collapse under any load imbalance.

2. Notice that the lower supports are completely useless. We know this because they meet at a right angle and there is no applied load – a quick mental force balance tells us that neither can possibly carry a load.

Members that are incapable of carrying a load are called zero-force members.

Recognizing zero-force members is very important when doing truss problems. You can save yourself a lot of work, and make an otherwise unsolvable problem solvable. Let us consider two common examples of zero force members:

#### Example 5.2: Colinear beams

Consider the following joint:
θ

\[ f_1 = \begin{bmatrix} -t_1 \\ 0 \\ 0 \end{bmatrix} \quad f_2 = \begin{bmatrix} t_2 \\ 0 \\ 0 \end{bmatrix} \quad f_3 = \begin{bmatrix} t_3 \cos \theta \\ t_3 \sin \theta \\ 0 \end{bmatrix} \]

Do a force balance at the joint:

\[
\sum f = \begin{bmatrix} -t_1 + t_2 + t_3 \cos \theta \\ t_3 \sin \theta \end{bmatrix} = 0
\] (5.15)

Force balance in the y direction implies that either \( t_3 \) is zero or \( \cos \theta \) is zero. But since \( \theta \) is given, it cannot be zero, so \( t_3 \) must be zero.

What happens if we apply a force to the joint?

\[ f_1 = \begin{bmatrix} -t_1 \\ 0 \\ 0 \end{bmatrix} \quad f_2 = \begin{bmatrix} t_2 \\ 0 \\ 0 \end{bmatrix} \quad f_3 = \begin{bmatrix} t_3 \cos \theta \\ t_3 \sin \theta \\ 0 \end{bmatrix} \]

Doing our force balance again we have

\[
\sum f = \begin{bmatrix} -t_1 + t_2 + t_3 \cos \theta + f_y \\ t_3 \sin \theta + f_y \end{bmatrix} = 0
\] (5.16)

Because there are now two forces in the y direction, we see that \( t_3 \) is not necessarily zero. Therefore, if a load is applied in this way, there are not necessarily any zero force members.

Thus, we conclude that if two members in a three-beam joint are colinear and the third is not, and there are no additional forces applied, the third beam must be a zero-force member.

**Example 5.3 : Free corner joint**

Consider the following joint:

\[ f_2 = \begin{bmatrix} t_2 \cos \theta \\ t_2 \sin \theta \\ 0 \end{bmatrix} \quad f_1 = \begin{bmatrix} t_1 \\ 0 \\ 0 \end{bmatrix} \]

Doing a force balance we have:

\[
\sum f = \begin{bmatrix} t_1 + t_2 \cos \theta \\ t_2 \sin \theta \end{bmatrix} = 0
\] (5.17)

Force balance in the y direction gives us that \( t_2 \) is zero, and then plugging that into the x direction we get that \( t_1 \) must be zero as well. Thus we conclude that both beams are zero force members.

What happens if we apply a force to the joint?

\[ f_2 = \begin{bmatrix} t_2 \cos \theta \\ t_2 \sin \theta \\ 0 \end{bmatrix} \quad f_1 = \begin{bmatrix} t_1 \\ 0 \\ 0 \end{bmatrix} \]
This changes our force balance:

\[
\sum f = \left[ t_1 + t_2 \cos \theta + f_x \right] = 0
\]

What can we say about our the tensions in our beams? Not much, unless it happens that one of the forces is colinear with the beam. Thus, we conclude that for a two-beam joint with non-colinear beams and no applied force, both beams must be zero force members.

**Health warning:** it is tempting to try to identify zero force members by looking at the members themselves and figuring out whether they are transmitting a load; however, this is very dangerous! Intuition frequently leads us the wrong way in problems of this type. The best thing to do when identifying zero force members is to look at the joints, do a mental force balance, and see if you can deduce why one or more of the tensions must be zero.

### 5.2.2 Stabilizing members

Consider the following example of a truss in equilibrium.

Upon inspection, we notice that this truss also has some serious problems.

1. We note that the forces and moments balance as long as the forces are exactly equal. However, the tiniest inequality in the horizontal forces vs. the vertical forces will cause the truss to collapse. This is called an **unstable truss**.

2. The truss can be stabilized by the addition of another member as shown above. The additional member will only carry a small stabilizing load, and so it does not have to be as strong. In our force analysis we'll see that the force in this member is zero; however, we recognize that it is necessary to stabilize the truss. This is called a **stabilizing member**.
Example 5.4: Warren Truss

Consider the following truss constructed of 11 beams. Beams \( ab, bc, cd, ef, fg \) are of length \( L \), and the beams \( ag, gb, bf, fc, ce, ed \) have length such that the angle between them and the other beams is \( \theta \). The truss is subjected to two loads:

![Truss Diagram]

Find the tension in the beams.

1. **Unknowns:** \( t_{ab}, t_{bc}, t_{cd}, t_{ag}, t_{bg}, t_{bf}, t_{cf}, t_{ce}, t_{de}, t_{ef}, f_{pivx}, f_{pivy}, f_{rolx} = 14 \) unknowns.

2. **Draw a FBD:**

   \[
   \begin{vmatrix}
   t_{fg} & -t_{fg} \\
   0 & 0 \\
   -t_{ag}n_1 & -t_{bg}n_2 \\
   t_{bg}n_2 & t_{bf}n_1 - t_{cf}n_2 \\
   t_{bf}n_1 & t_{bc}n_2 - t_{ef}n_2 \\
   -t_{de}n_1 & -t_{de}n_2 \\
   t_{de}n_2 & t_{de}n_1 \\
   \end{vmatrix}
   \]

   where

   \[
   n_1 = \begin{bmatrix}
   \cos \theta \\
   \sin \theta \\
   \end{bmatrix}, \quad n_2 = \begin{bmatrix}
   -\cos \theta \\
   \sin \theta \\
   \end{bmatrix}
   \]

3. **Equations:** we have seven joints so \([\text{seven joints}] \times [\text{two equations per joint}] = 14 \) equations.

   This is going to be a big hassle, because none of these equations will allow us to solve for a tension immediately. But...we can solve for \( f_{pivx}, f_{pivy}, f_{rolx} \) right away if we just treat the truss as a rigid body.

   Let's take the moment of the two forces and the roller reaction about the pivot.

   \[
   \begin{aligned}
   r_{ab} &= \begin{bmatrix}
   L \\
   0 \\
   0 \\
   \end{bmatrix} \\
   r_{ac} &= \begin{bmatrix}
   2L \\
   0 \\
   0 \\
   \end{bmatrix} \\
   r_{ad} &= \begin{bmatrix}
   3L \\
   0 \\
   0 \\
   \end{bmatrix}
   \end{aligned}
   \]

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so the moments are

\[
M_{ab} = r_{ab} \times f = \begin{bmatrix} 0 \\ 0 \\ -Lf_1 \end{bmatrix} \quad M_{ac} = r_{ac} \times f = \begin{bmatrix} 0 \\ 0 \\ -2Lf_2 \end{bmatrix} \quad M_{ad} = r_{ad} \times f_{rol}^d = \begin{bmatrix} 0 \\ 0 \\ 3Lf_{rol,y}^d \end{bmatrix}
\] (5.21)

by moment equilibrium, we get that

\[-Lf_1 - 2Lf_2 + 3Lf_{rol,y}^d = 0 \implies f_{rol,y}^d = \frac{L}{3} f_1 + \frac{2}{3} f_2 \] (5.22)

by force equilibrium of the truss, we have

\[
\begin{bmatrix} f_{piv,x}^p \\ f_{piv,y}^p \\ 0 \end{bmatrix} + \begin{bmatrix} -f_1 \\ -f_2 \\ 0 \end{bmatrix} - \begin{bmatrix} f_1 \\ f_2 \\ 0 \end{bmatrix} = 0 \implies f_{piv,x}^p = 0 \quad \text{and} \quad f_{piv,y}^p = \frac{2}{3} f_1 + \frac{1}{3} f_2
\] (5.23)

point d:

\[
\begin{bmatrix} t_{cd} \\ f_{rol,y}^d + t_{de} \sin \theta \end{bmatrix} = 0 \implies t_{de} = -\frac{f_1 + 2f_2}{3 \sin \theta} \quad \text{and} \quad t_{cd} = \frac{f_1 + 2f_2}{3 \tan \theta}
\] (5.24)

point e:

\[
\begin{bmatrix} -t_{ef} \\ -t_{ce} \cos \theta + t_{de} \cos \theta \\ -t_{ce} \sin \theta - t_{de} \sin \theta \end{bmatrix} = 0 \implies t_{ce} = \frac{f_1 + 2f_2}{3 \sin \theta} \quad \text{and} \quad t_{ef} = -\frac{2f_1 + 4f_2}{3 \tan \theta}
\] (5.25)

point c:

\[
\begin{bmatrix} -t_{bc} \\ -t_{cf} \cos \theta + t_{ce} \cos \theta + t_{cd} \\ t_{cf} \sin \theta + t_{ce} \sin \theta - f_2 \end{bmatrix} = 0 \implies t_{cf} = \frac{f_2 - f_1}{3 \sin \theta} \quad \text{and} \quad t_{bc} = \frac{f_1 + f_2}{\tan \theta}
\] (5.26)

By symmetry, we would see that \( t_{bf} = t_{cf}, t_{gf} = t_{ef} \) and so on.

(5) Sanity check: what if \( f_1 = f_2 = f \)? Then

\[ f_{bc} = \frac{2f}{\tan \theta} \quad \text{and} \quad f_{ef} = -\frac{2f}{\tan \theta}
\] (5.27)

which implies that even though the force is in the vertical direction, the load is all carried by the horizontal members!

What happens when \( \theta \) increases? The beam becomes “fatter” and the tensions in the top and bottom members decrease! This indicates that fatter beams are stronger. This sort of effect will pop up a lot when you study strength of materials.

**Example 5.5**

A truss is constructed of four beams with length \( L \) and a crossbeam as shown in the following figure.
Determine the magnitude of the forces in each member and the reactions at the roller and pivot.

(1) Determine unknowns: \( t_{ab}, t_{bc}, t_{cd}, t_{ad}, t_{bd}, f_{pc}, f_{pc}, f_{roly} = [8 \text{ unknowns}] \)

(2) Draw FBD:

\[
\begin{bmatrix}
-\frac{t_{ad} \cos \theta}{2} \\
\frac{-t_{ad} \sin \theta}{2}
\end{bmatrix}
\]

(3-4) Write equations and solve.

Check: how many equations do we have? \([4 \text{ nodes}] \times [2 \text{ equilibrium equations per node}] = [8 \text{ equations}]\)

node a:

\[
\sum f = \begin{bmatrix}
-t_{ab} + t_{ad} \cos \theta \\
-f - t_{ad} \sin \theta
\end{bmatrix} = 0 \implies t_{ad} = -\frac{f}{\sin \theta} \implies t_{ab} = -\frac{f}{\tan \theta}\]

node b:

\[
\sum f = \begin{bmatrix}
t_{bc} + t_{bd} \cos \theta + t_{bd} \cos(\theta/2) \\
-t_{bc} \sin \theta - t_{bd} \sin(\theta/2)
\end{bmatrix} = 0
\]

or

\[
\begin{bmatrix}
\cos \theta & \cos(\theta/2) \\
\sin \theta & \sin(\theta/2)
\end{bmatrix}
\begin{bmatrix}
t_{bc} \\
t_{bd}
\end{bmatrix}
= \begin{bmatrix}
f/\tan \theta \\
0
\end{bmatrix}
\]

The determinant is \(\sin(\theta/2) \cos \theta - \cos(\theta/2) \sin \theta = \sin(\theta/2 - \theta) = \sin(-\theta/2) = -\sin(\theta/2)\). Use Cramer's rule:

\[
t_{bc} = -\frac{1}{\sin(\theta/2)} \det \begin{bmatrix}
f/\tan \theta & \cos(\theta/2) \\
0 & \sin(\theta/2)
\end{bmatrix} = -\frac{f}{\tan \theta}
\]
\[ t_{bd} = -\frac{1}{\sin \theta/2} \det \begin{bmatrix} \cos \theta & f/\tan \theta \\ \sin \theta & 0 \end{bmatrix} = \frac{f \sin \theta/\tan \theta}{\sin(\theta/2)} = \frac{f \cos \theta}{\sin(\theta/2)} \] (5.33)

node d:
\[ \sum f = \begin{bmatrix} -t_{ad} \cos \theta - t_{bd} \cos(\theta/2) - t_{cd} \\ t_{ad} \sin \theta + t_{bd} \sin(\theta/2) + f_{roly} \end{bmatrix} = 0 \] (5.34)
\[ t_{cd} = -\left(-\frac{f}{\sin \theta}\right) \cos \theta - \left(\frac{f \cos \theta}{\sin(\theta/2)}\right) \cos(\theta/2) = f \left(\frac{1}{\tan \theta} - \frac{\cos \theta}{\tan(\theta/2)}\right) \] (5.35)
\[ f_{roly} = -t_{ad} \sin \theta - t_{bd} \sin(\theta/2) = f - f \cos \theta = f(1 - \cos \theta) \] (5.36)

node c:
\[ \sum f = \begin{bmatrix} f_{cpx} - t_{bc} \cos \theta + t_{cd} \\ f_{cpy} + t_{bc} \sin \theta \end{bmatrix} = 0 \] (5.37)
\[ \Rightarrow f_{cpx} = t_{bc} \cos \theta - t_{cd} = -\frac{f \cos \theta}{\tan \theta} - \frac{f}{\tan \theta} + \frac{f \cos \theta}{\tan(\theta/2)} = 0 \] (5.38)
\[ \Rightarrow f_{cpy} = f \cos \theta \] (5.39)

A simple check shows that force balance is satisfied:
\[ \sum f = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} f(1 - \cos \theta) \\ -f \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \checkmark \] (5.40)

and moment balance about the pivot is satisfied as well:
\[ \sum M = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ Lf(1 - \cos \theta) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \checkmark \] (5.41)

Note that we recovered the rigid body solution without having to do any analysis with moments.

5.3 Stability

In the previous lecture, the issue of stability was alluded to; let's explore that a little further. Consider the following truss:

Let's temporarily treat the truss as a rigid body. How many equations do we have? Two force equilibrium equations, one moment equilibrium equation. How many unknowns? Three, this is the most number of constraint unknowns we can have for a rigid body.

How many unknowns do we have in the truss? Three: the tension in each member. How many equations? Six: two force equilibrium equations for each. We have a total of six unknowns and six equations, so the system is solvable.

Now consider the following truss:
How many unknowns? [3 constraints] + [4 members] = [7 unknowns]
How many equations? [4 nodes] x [2 equations per node] = [8 unknowns]
This system is not solvable, and is unstable.
How can we fix the system? We need another unknown. We could introduce this by adding a constraint:

The figure on the left is suitably stabilized and solvable.
The figure on the right has the same number of equations and unknowns, but is it solvable? What will happen if I apply a load? If I write the equilibrium equations here I will get something like this:

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
t_{ab} \\
t_{bc} \\
t_{cd} \\
t_{da} \\
t_a \\
t_b \\
t_c \\
t_d \\
\end{bmatrix} = \begin{bmatrix} f_a \\
f_b \\
f_c \\
f_d \end{bmatrix}
$$

What is the determinant of this matrix? Zero, so it is not solvable. But if I apply the constraint to node c instead, I get a determinant of -1.

## 5.4 Method of sections

In the last section, we used the **method of joints** to evaluate the forces in the members of a truss.
What if we want to find the force in just one member? Using the method of joints, this is usually not possible. The **method of sections** allows us to find the tensions in any member we choose with a minimal amount of work.
We will illustrate this with an example:

**Example 5.6**

Consider the following Warren truss made with members that all have length \(L\).

Find the tension in element de.
First, let us compute the reactions using rigid body equilibrium.
To get $r_{h}$, compute the moment about the pivot.

\[
\begin{bmatrix}
    f_{pivx}^o \\
    f_{pivy}^o
\end{bmatrix}
\]

\[
\begin{bmatrix}
    0 \\
    r_{h}
\end{bmatrix}
\]

so moment equilibrium gives us

\[
\begin{bmatrix}
    0 \\
    7Lr_{h} - Lf - 2Lf - 3Lf - 4Lf - 5Lf - 6Lf
\end{bmatrix} = 0 \implies 7r_{h} = 21f \implies r_{h} = 3f
\]  

(5.45)

Using force equilibrium we get

\[
\begin{bmatrix}
    f_{pivx}^o f_{pivy}^o - 6f + 3f
\end{bmatrix} = 0 \implies f_{piv}^o = \begin{bmatrix}
    0 \\
    3f
\end{bmatrix}
\]  

(5.46)

With the old technology, we would have to a whole lot of work before we finally make it to member $de$.

Let us use the method of sections. We will cleverly introduce a cut as shown in the following diagram. We know that the selected portion must be in equilibrium; therefore we can solve the equilibrium equations to obtain the reactions at the boundary. As always, draw a free body diagram:

\[
\begin{bmatrix}
    0 \\
    3f
\end{bmatrix}
\]

\[
\begin{bmatrix}
    -f \\
    -f
\end{bmatrix}
\]

\[
\begin{bmatrix}
    -f \\
    -f
\end{bmatrix}
\]

\[
\begin{bmatrix}
    0 \\
    3f
\end{bmatrix}
\]

\[
\begin{bmatrix}
    t_{de} \\
    t_{ke}
\end{bmatrix}
\]

\[
\begin{bmatrix}
    t_{ke} \cos \theta \\
    t_{ke} \sin \theta
\end{bmatrix}
\]

We want $t_{de}$. How can we obtain it? We can take moment equilibrium about point k:

\[
\begin{bmatrix}
    -4L \\
    0 \\
    0
\end{bmatrix}
\]

\[
\begin{bmatrix}
    -3L \\
    0 \\
    0
\end{bmatrix}
\]

\[
\begin{bmatrix}
    -2L \\
    0 \\
    0
\end{bmatrix}
\]

\[
\begin{bmatrix}
    -L \\
    0 \\
    0
\end{bmatrix}
\]

\[
\begin{bmatrix}
    0 \\
    L \sqrt{3}/2 \\
    0
\end{bmatrix}
\]

\[
\begin{bmatrix}
    0 \\
    0 \\
    0
\end{bmatrix}
\]

\[
\begin{bmatrix}
    0 \\
    0 \\
    -t_{de} L \sqrt{3}/2
\end{bmatrix}
\]  

(5.47)  

(5.48)
From moment equilibrium we get

\[-12Lf + 3Lf + 2Lf + Lf - t_{de}L\sqrt{3}/2 = 0 \implies t_{de} = -\frac{12}{\sqrt{3}} \frac{f}{L}\]  

(5.49)

Way easier than the method of joints!

---

**Example 5.7**

Consider the following truss:

![Truss Diagram](image)

Find the magnitude of the force in members \(jk, kc, bc\).

First, let’s compute the magnitude of the force at the pivot and the roller. Free body diagram:

![Free Body Diagram](image)

Take moment equilibrium about \((a)\):

\[
\begin{align*}
\mathbf{r}^j &= \begin{bmatrix} L \\ H \\ 0 \end{bmatrix} & \mathbf{r}^k &= \begin{bmatrix} 2L \\ 2H \\ 0 \end{bmatrix} & \mathbf{r}^l &= \begin{bmatrix} 3L \\ 3H \\ 0 \end{bmatrix} & \mathbf{r}^i &= \begin{bmatrix} 4L \\ 2H \\ 0 \end{bmatrix} & \mathbf{r}^h &= \begin{bmatrix} 5L \\ 1H \\ 0 \end{bmatrix} & \mathbf{r}^g &= \begin{bmatrix} 6L \\ 0 \\ 0 \end{bmatrix} \\
\mathbf{M}^j &= \begin{bmatrix} 0 \\ 0 \\ -fl \end{bmatrix} & \mathbf{M}^k &= \begin{bmatrix} 0 \\ 0 \\ -2fl \end{bmatrix} & \mathbf{M}^l &= \begin{bmatrix} 0 \\ 0 \\ -3fl \end{bmatrix} & \mathbf{M}^i &= \begin{bmatrix} 0 \\ 0 \\ -4fl \end{bmatrix} & \mathbf{M}^h &= \begin{bmatrix} 0 \\ 0 \\ -5fl \end{bmatrix} & \mathbf{M}^g &= \begin{bmatrix} 0 \\ 0 \\ 6Lf_{roly} \end{bmatrix}
\end{align*}
\]  

(5.50)

(5.51)

z-moment equilibrium gives:

\[6Lf_{roly}^g = fl + 2fl + 3fl + 4fl + 5fl = 15fl \implies f_{roly}^g = \frac{5}{2}f\]  

(5.52)
and force equilibrium gives:

\[
\begin{bmatrix}
\mathbf{f}_{\text{piv}}^a \\
\mathbf{f}_{\text{pivy}}^a + 5\mathbf{f} - \frac{5}{2}\mathbf{f}
\end{bmatrix} = \mathbf{0} \implies \mathbf{f}_{\text{rotx}}^a = \mathbf{0} \quad \mathbf{f}_{\text{roty}}^a = \frac{5}{2}\mathbf{f}
\] (5.53)

Now, let us try to cleverly pick a cut. (use what is shown above)

![Diagram](image)

Do rigid body analysis: take the moment about point (c)

\[
\begin{align*}
\mathbf{r}^a &= \begin{bmatrix} -2L \\ 0 \\ 0 \end{bmatrix} & \mathbf{r}' &= \begin{bmatrix} -L \\ H \\ 0 \end{bmatrix} & \mathbf{r}^{jk} &= \begin{bmatrix} 0 \\ 2H \\ 0 \end{bmatrix} \\
\mathbf{M}_{\text{piv}}^a &= \begin{bmatrix} 0 \\ 0 \\ -5Lf \end{bmatrix} & \mathbf{M}^b &= \begin{bmatrix} 0 \\ 0 \\ fL \end{bmatrix} & \mathbf{M}^{jk} &= \begin{bmatrix} 0 \\ -2t_{jk}HL/\sqrt{L^2 + H^2} \end{bmatrix}
\end{align*}
\] (5.54)

Moment equilibrium gives:

\[
t_{jk} = -2f\sqrt{(L/H)^2 + 1}
\] (5.56)

The result is in compression as we expect.

(Interesting note: we've seen with other trusses how the end result does not depend on length. Notice here that the result depends only on the dimensionless quantity \(L/H\), that is, the aspect ratio of the truss.)
6 Internal Forces

6.1 Internal forces in beams

In the previous sections, we treated rigid bodies that were connected to each other in various ways; most frequently, with pins and fixed joints. In the last homework, we saw that we can treat a continuous body as two separate bodies connected via a fixed joint. We are going to continue that line of thinking here.

6.1.1 Method of sections for beams in 2D

A beam is loaded as follows under a variety of distributed loads, points loads, and moments. We will consider a section of this beam, as outlined by the box:

![Diagram of beam with forces and moments marked.]

We can draw a free body diagram for this section by treating both ends as fixed joints. The connecting forces and moments are called internal forces and moments for the beam. In this section, we are interested in finding these internal forces.

6.1.2 Notation and conventions

At this point we will introduce a slightly different convention for keeping track of our internal forces. This will simplify our analysis and will be more closely aligned with convention. Let us redraw the diagram we have above:

![Redrawn diagram with new notation for forces and moments.]

In this diagram:

- $N$ is the x component of the force reaction. It is called the normal force.
- $V$ is the y component of the force reaction. It is called the shear force.
- $M$ is the z component of the moment reaction. It is called the bending moment.

Notice that I did not give anything a negative sign. The directionality is implied by the diagram, and we keep track of it as follows:

- $N$ acts in the negative-x on the left, positive-x on the right.
- $V$ acts in the positive-y on the left, negative-y on the right.
- $M$ acts in the negative-z on the left, positive-z on the right.

Some notes:
If it were up to me, we would swap the shear convention. But for better or worse, this is what everyone uses so we'll stick with it.

This convention guarantees that the forces and moments always sum to zero at each cut. Therefore, we don't generally have to worry about it.

6.1.3 A note about couple moments

- Single rigid body: couple moment acts equally regardless of location.
- Multiple rigid bodies: couple moment acts on one member only (regardless of location).
- Section of a beam: couple moment acts on the section only if it is located in the section. 

/> the location of a moment is important for internal force analysis.

6.1.4 Solution strategy

(A) Rigid body analysis to compute all reaction forces, moments, etc. – see previous sections.

(B) Internal force analysis:

1. Determine unknowns
2. Pick a section and draw its free body diagram
3. Write force and moment equilibrium equations:
4. Solve
5. Sanity check
6. Substitute numbers

6.1.5 Examples

Example 6.1

Consider the following beam:

Find the internal shear force, normal force, and bending moment at the center.
Part (A): rigid body analysis. Fortunately, this is a super easy problem to solve, and we can get all of our reaction forces really quickly. We need to solve for \( f_{pivx} \), \( f_{pivy} \), \( f_{roly} \). Computing moments about the pivot, we get:

\[
r_f = \begin{bmatrix} L \\ 0 \\ 0 \end{bmatrix} \quad f = \begin{bmatrix} 0 \\ -f \\ 0 \end{bmatrix} \quad M_f = \begin{bmatrix} 0 \\ 0 \\ -fL \end{bmatrix}
\] (6.1)

\[
r_{rol} = \begin{bmatrix} 3L \\ 0 \\ 0 \end{bmatrix} \quad f_{rol} = \begin{bmatrix} 0 \\ f_{roly} \\ 0 \end{bmatrix} \quad M_{rol} = \begin{bmatrix} 0 \\ 0 \\ 3Lf_{roly} \end{bmatrix}
\] (6.2)
Moment equilibrium in the z direction gives:
\[ \sum M = \begin{bmatrix} 0 \\ 0 \\ -fL \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 3L \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ M \end{bmatrix} = 0 \implies f_{roly} = \frac{f}{3} \frac{M}{3L} \quad (6.3) \]

Force equilibrium in the x and y directions gives:
\[ f_{pivx} = 0 \quad (6.4) \]
\[ f_{pivy} = f - \frac{f}{3} + \frac{M}{3L} = \frac{2f}{3} + \frac{M}{3L} \quad (6.5) \]

For part (B):

1. Determine unknowns: we want to find \( N, V, M \) at the center.

2. Cut the beam exactly in half:

\[ \begin{bmatrix} 0 \\ -f \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ -f \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 2f/3 + M/3L \\ 0 \end{bmatrix} = \begin{bmatrix} N_0 \\ V_0 \\ M_0 \end{bmatrix} \]

3. Write equations and solve.

   Force equilibrium:
\[ \sum f = \begin{bmatrix} 2f/3 + M/3L \\ -f \\ 0 \end{bmatrix} + \begin{bmatrix} N \\ V \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (6.6) \]

   x-equilibrium gives
\[ N_0 = 0 \quad (6.7) \]

   y-equilibrium gives
\[ \frac{2f}{3} + \frac{M}{3L} - f - V_0 = 0 \implies V_0 = \frac{M}{3L} - \frac{f}{3} \quad (6.8) \]

Take moments about the cut:
\[ r_{piv} = \begin{bmatrix} -3L/2 \\ 0 \\ 0 \end{bmatrix} \quad (6.9) \]
\[ r_f = \begin{bmatrix} -L/2 \\ 0 \\ 0 \end{bmatrix} \]
\[ M_{piv} = \begin{bmatrix} 0 \\ 0 \\ -(3L/2)(2f/3 + M/3L) \end{bmatrix} \quad (6.10) \]
\[ M_f = \begin{bmatrix} 0 \\ 0 \\ fL/2 \end{bmatrix} \]

Moment equilibrium gives:
\[ \sum M = \begin{bmatrix} 0 \\ -(3L/2)(2f/3 + M/3L) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ fL/2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ M_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (6.11) \]
Z equilibrium gives

\[ M_0 = \frac{Lf}{2} + \frac{M}{2} \]  

(6.12)

As a sanity check, let’s try using the other half of the beam:

\[ f_{rol} = \begin{bmatrix} 0 \\ 0 \\ f/3 - M/3L \end{bmatrix} \]

x-force equilibrium gives:

\[ N_0 = 0 \checkmark \]  

(6.13)

y-force equilibrium gives:

\[ V_0 + \frac{f}{3} - \frac{M}{3L} = 0 \implies V_0 = \frac{M}{3L} - \frac{f}{3} \]  

(6.14)

moment equilibrium about the cut gives:

\[ \frac{3L}{2} \left( \frac{f}{3} - \frac{M}{3L} \right) + M - M_0 = 0 \implies M_0 = \frac{M}{2} + \frac{Lf}{2} \checkmark \]  

(6.15)

### 6.2 Shear-moment diagrams

We have discussed how to compute the shear force and moment at a particular point. However, when analyzing beams, it is frequently useful to evaluate the internal forces about all of the points. These are visualized using plots, called shear-moment diagrams.

(Note: the normal force usually turns out to be zero, so we will generally neglect it.)

This is best illustrated with an example:

**Example 6.2**

Let’s consider the following beam:

We’ve already learned how to compute the reaction at the fixed joint, and we even know how to find the internal forces at a point in this beam. We want to find the shear and the moment at all points in the beam.

Let us consider the region defined by \( 0 \leq x < L \):
Our equilibrium equations give us:

\[ V(x) = 0 \quad M(x) = 0 \]  

(6.16)

Easy enough! We plot these quantities (except \( N(x) \)...it's not going to be very interesting) as shown.

Now let's consider the region \( L < x < 2L \): draw a FBD.

Let us do equilibrium. Force equilibrium in the \( y \) direction gives

\[ -f_a - V(x) = 0 \implies V(x) = -f_a, \quad L < x < 2L \]  

(6.17)

Now let us consider moment equilibrium in the \( z \) direction. Let us sum the moments about the cut. This gives us:

\[ \sum M_z = (f_a)(x - L) + M(x) = 0 \implies M(x) = -f_a(x - L), \quad L < x < 2L \]  

(6.18)

We can plot this on our shear and moment diagram:

Now, let's consider the region \( 2L < x < 3L \). Draw a free body diagram:

Equilibrium in the \( y \) direction:

\[ -f_a - V(x) = 0 \implies V(x) = -f_a, \quad 2L < x < 3L \]  

(6.19)
Moment equilibrium:

\[ f_a(x - L) + M_a + M(x) = 0 \implies M(x) = -f_a(x - L) - M_a \quad 2L < x < 3L \]  

(6.20)

Plotting this on our FBD:

Equilibrium in the y direction gives:

\[ -f_a - f_b - V(x) = 0 \implies V(x) = -f_a - f_b, \quad 3L < x < 4L \]  

(6.21)

Equilibrium of moments gives:

\[ f_a(x - L) + M_a + f_b(x - 3L) + M(x) = 0 \implies M(x) = -f_a(x - L) - M_a - f_b(x - 3L) \quad 3L < x < 4L \]  

(6.22)

Plotting:

What about the reaction at the wall? We know that the force and moments will go to zero. On our diagram, then, we draw:

**Distributed loads revisited**

There has been some confusion on computing the total force and moment from a distributed load.
w(x) \left( \frac{\text{force}}{\text{length}} \right)
\frac{dx}{\text{length}}
\frac{df}{\text{length}} = w(x)dx \left( \frac{\text{force}}{\text{length}} \right)
\frac{dm}{\text{length}} = r(x)w(x)dx \left( \frac{\text{force}}{\text{length}} \right)

Things to remember:

- $w(x)$ is our force per unit length – it only makes sense as a force when multiplied by something with units of length.
- $df = w(x)dx$ is a differential force element – it has units of force, and must be integrated to give a finite value, e.g.
  \[
  \int_0^L df = \int_0^L w(x)dx
  \]  (6.23)
- $dm = r(x)df = r(x)w(x)dx$ is a differential moment about a point, where $r(x)$ is the distance.
  (Note that we are working in 2D, so we drop the vector notation and take all moments to be in the $z$-direction implicitly). We must integrate $dm$ to get the total moment, e.g.
\[
\int_0^L dm = \int_0^L r(x)w(x)dx
\]  (6.24)

In previous cases, we took $r(x)$ to just be $x$, but we need to be careful about this in the future.

**Example 6.3**

Draw a shear-moment diagram for the following beam:

\[
w(x) = -w_0\]

Last time, in the previous example, we did not consider the reactions. We could get away with that because one end was free; however, this time, we need to go ahead and find the reactions.

As an example of taking the moment about a different point, let us take the moment about the roller instead of the pivot. The total moment of the pivot and the distributed load about the roller will give us
\[
\sum M_z = -f_{pivy}L + \int_0^L w(x)(L-x)dx = 0
\]  (6.25)
\[
\Rightarrow f_{pivy} = \frac{1}{L} \int_0^L w_0(L-x)dx = \frac{1}{L} \int_0^L w_0Ldx - \int_0^L \frac{1}{L} w_0x dx = w_0L - \frac{1}{2L} w_0 \frac{L^2}{2} = \frac{w_0L}{2}
\]  (6.26)

Force equilibrium in the $x$ and $y$ directions gives us
\[
\sum f_x = f_{pivx} = 0
\]  (6.27)
\[
\sum f_y = \frac{w_0L}{2} + f_{roly} - \int_0^L w_0dx = f_{roly} - w_0L + \frac{w_0L}{2} = 0 \quad \Rightarrow \quad f_{roly} = \frac{w_0L}{2}
\]  (6.28)
Now, let us take a cut of the beam to construct a free body diagram and solve for \( V(x) \), \( M(x) \): Force balance in the y direction gives us:

\[
\sum f_y = \frac{w_0L}{2} - V(x) - \int_0^x w_0\delta \, d\delta = 0
\]

\[
\implies V(x) = \frac{w_0L}{2} - w_0\delta \bigg|_{0}^{x} = \frac{1}{2}w_0L - w_0x
\]  

(6.29)  

(6.30)

Finally, do moment balance in the z direction about the cut. Remember, we have to be careful about how we choose our distance vector.

\[
\sum M_z = -\left(\frac{w_0L}{2}\right)(x) + \int_0^x (w(x))(x - \delta)\, d\delta + M(x)
\]

\[
\implies M(x) = \frac{w_0Lx}{2} - w_0\int_0^x (x - \delta)\, d\delta = \frac{w_0Lx}{2} - w_0x\int_0^x \delta \, d\delta + w_0\int_0^x \delta \, d\delta
\]

\[
= \frac{1}{2}w_0Lx - \frac{1}{2}w_0x^2
\]  

(6.31)  

(6.32)  

(6.33)

Let’s plot the result:

![Plot of shear vs distance](image1.png)  

![Plot of moment vs distance](image2.png)

Plot of shear vs distance  

Plot of moment vs distance
6.3 Elementary beam theory

6.3.1 Relationship between loading, shear, and moment

It turns out that there are some neat relationships between loading, shear, and moment that make it much, much easier to compute shear-moment diagrams for complicated systems. Let us consider a small piece of a beam that is subjected to some distributed load \( w(x) \). What are the shear forces and moments?

\[
\begin{align*}
V(x) & \quad w(x) \\
N(x) & \quad M(x + \Delta x) \\
M(x) & \quad N(x + \Delta x) \\
\Delta x & \quad V(x + \Delta x)
\end{align*}
\]

First, let us do force equilibrium in the x direction.

\[
\sum F_x = -N(x) + N(x + \Delta x) = 0 \quad (6.34)
\]

Dividing both sides by \( \Delta x \) and letting \( \Delta x \to 0 \) we get

\[
\lim_{\Delta x \to 0} \left( \frac{N(x + \Delta x) - N(x)}{\Delta x} \right) = \frac{dN(x)}{dx} = 0 \quad (6.35)
\]

As long as there are no lateral forces applied, we get that the normal force in the beam does not change. Now, let us do force equilibrium in the y direction:

\[
\sum F_y = V(x) - V(x + \Delta x) + w(x)\Delta x = 0 \quad (6.36)
\]

We can pull the exact same trick! Let us divide both sides by \( \Delta x \):

\[
\frac{V(x + \Delta x) - V(x)}{\Delta x} = w(x) \quad (6.37)
\]

now, let \( \Delta x \to 0 \)

\[
\frac{dV(x)}{dx} = w(x) \quad (6.38)
\]

This is a very useful result, because it means that we can obtain \( V(x) \) just by integrating \( w(x) \)!

Let’s see if we get anything interesting by taking the moments. Let us take the moments about the right hand side (assuming a uniform load)

\[
\sum M_x = -V(x)\Delta x - (w(x)\Delta x) \left( \frac{\Delta x}{2} \right) + M(x + \Delta x) - M(x) = 0 \quad (6.39)
\]

This is not as pretty! But maybe we can simplify a few things. Let us try doing the same thing we did before and divide by \( \Delta x \):

\[
\frac{M(x + \Delta x) - M(x)}{\Delta x} - V(x) - \frac{1}{2} w(x)\Delta x = 0 \quad (6.40)
\]
Now, let $\Delta x \to 0$ we get

$$\frac{dM(x)}{dx} = V(x)$$  \hspace{1cm} (6.41)

Another interesting result! This means that we can compute $M(x)$ simply by integrating $V(x)$ or by integrating $w(x)$ twice.

**Example 6.4**

Let’s verify this result with the example that we did before. We had the result:

$$w(x) = -w_0 \hspace{1cm} V(x) = \frac{1}{2} w_0 L - w_0 x M(x) = \frac{1}{2} w_0 L x - \frac{1}{2} w_0 x^2$$  \hspace{1cm} (6.42)

Does this work? Sure enough,

$$\frac{dM}{dx} = \frac{1}{2} w_0 L (1) - \frac{1}{2} w_0 (2x) = \frac{1}{2} w_0 L - w_0 x = V(x)$$  \hspace{1cm} (6.43)

Differentiating again,

$$\frac{dV}{dx} = 0 - w_0 (1) = -w_0$$  \hspace{1cm} (6.44)

**Example 6.5**

Compute the shear and moment diagrams for the following

$$w(x) = w_0 \sin \left( \frac{\pi x}{L} \right)$$

Let us draw a free body diagram:

Note that I have replaced the familiar $f_{fix}, M_{fix}$ with $M(0)$ and $V(0)$. Remember that this is just a change in notation.

Let us compute these values. Summing forces in the y direction we have

$$\sum f_y = V(0) + \int_0^L w_0 \sin \left( \frac{\pi x}{L} \right) dx = V(0) + \left. \frac{w_0 L}{\pi} \cos \left( \frac{\pi x}{L} \right) \right|_0^L = V(0) + \frac{2w_0 L}{\pi} = 0$$  \hspace{1cm} (6.45)

$$\implies V(0) = -\frac{2w_0 L}{\pi}$$  \hspace{1cm} (6.46)
(where \( V(0) = f_{fix,y} \)). Now compute the moment. We know by symmetry that the effective force will act in the center of the beam, so we get

\[
\sum M_z = -M(0) + \frac{2w_0L}{\pi} \left( \frac{L}{2} \right) = 0 \implies M(0) = \frac{w_0L^2}{\pi}
\]  
(6.47)

Now, let us compute the shear force. We can do this simply by integrating:

\[
V(x) = \int w(x)dx = \int w_0 \sin(\pi x/L)dx = -\frac{w_0L}{\pi} \cos(\pi x/L) + C
\]  
(6.48)

We can solve for our constant using what we know about \( V(0) \):

\[
V(0) = -\frac{w_0L}{\pi} + C \implies -2\frac{w_0L}{\pi} \implies C = -\frac{w_0L}{\pi}
\]  
(6.49)

So we have

\[
V(x) = -\frac{w_0L}{\pi} (\cos(\pi x/L) + 1)
\]  
(6.50)

Now we can compute our moment by integrating:

\[
M(x) = \int V(x)dx = -\int \frac{w_0L}{\pi} (\cos(\pi x/L) + 1)dx = -\frac{w_0L}{\pi} \left( \frac{L}{\pi} \sin(\pi x/L) + x \right) + D
\]  
(6.51)

We can solve for \( D \) using what we know about \( M(0) \):

\[
M(0) = D = \frac{w_0L^2}{\pi}
\]  
(6.53)

so our moment is:

\[
M(x) = \frac{w_0L(L-x)}{\pi} - \frac{w_0L^2}{\pi^2} \sin(\pi x/L)
\]  
(6.54)

What should the shear and moment be at the free end? Zero, because there are no supports. Let’s verify this:

\[
V(L) = -\frac{w_0L}{\pi} (\cos(\pi) + 1) = -\frac{w_0L}{\pi} (-1 + 1) = 0 \checkmark
\]  
(6.55)

\[
M(L) = \frac{w_0L(L-L)}{\pi} - \frac{w_0L^2}{\pi^2} \sin(\pi) = 0 \checkmark
\]  
(6.56)

Plotting these curves, we get:
6.3.2 Boundary conditions

Boundary conditions are important to check when doing free body diagrams. They are often useful for checking you answer. Four types of boundary conditions are:

**Free end** Let's begin by considering and end of a beam where there are no constraints (or forces) acting at the end. We might draw a free end in the following way:

A free end cannot sustain a force or a moment. (Sound familiar? Freedom to move/rotate = inability to sustain force/moment).

\[
V(x_{\text{free end}}) = 0 \quad M(x_{\text{free end}}) = 0
\]  

(6.57)

**Pivot/roller end** Simply supported beams are frequently supported with pins or rollers. We might draw a pinned/roller end of a beam in the following way.

Pivots and rollers constrain the vertical motion of the beam, but they allow free rotation. Therefore our boundary conditions are

\[
V(x_{\text{pinned end}}) \neq 0 \quad M(x_{\text{pinned end}}) = 0
\]  

(6.58)

For a pivot, motion is additionally constrained in the x direction, so we note that \(N(x_{\text{pivot end}}) \neq 0\).

**Fixed rotation** This type of boundary condition doesn't come up very often, but it is an interesting example of a different type of boundary condition. We might draw it like this:
Note that it allows free vertical motion while also constraining rotation. The boundary conditions will be

\[ V(x_{\text{roller}}) = 0 \quad M(x_{\text{roller}}) \neq 0 \]  

(And we also note that \( N(x_{\text{roller}}) \neq 0 \) since lateral motion is constrained as well.)

**Fixed end**

Finally, consider a beam that is fixed at one end, that we might draw as:

As usual a fixed end constrains all of the degrees of freedom of our system, so we know that

\[ V(x_{\text{fix}}) \neq 0 \quad M(x_{\text{fix}}) \neq 0 \]  

(6.60)

And, of course, we note that \( N(x_{\text{fix}}) \neq 0 \) as well.

### 6.3.3 Dirac delta distributions

We have established a differential relationship between applied load, shear, and moment. But what happens if we have a point load? How do we integrate that thing? To do this, we need **Dirac delta distributions**.

(Dirac distributions are often incorrectly called “delta functions” by engineers. This is a misnomer—is is not, strictly speaking, a function.)

Consider the following box function.

\[ f(x) = \begin{cases} \frac{1}{L} & -L/2 < x < L/2 \\ 0 & \text{else} \end{cases} \]  

(6.61)

What happens if I integrate this curve?

\[ \int_{-\infty}^{\infty} f(x) = \int_{-L/2}^{L/2} \left( \frac{1}{L} \right) dx = \frac{L/2 - (-L/2)}{L} = 1 \]  

(6.62)

I will get a value of 1. What happens if I make \( L \) really, really small?

\[ \lim_{L \to 0} f(x) \]  

(6.63)
The value of \( f(x) \) goes to infinity at 0 and 0 elsewhere...but we still have that

\[
\lim_{L \to 0} \int_{-\infty}^{\infty} f(x) = 1
\]  

(6.64)

This is called a Dirac distribution, and we generally write them as

\[
\delta(x)
\]  

(6.65)

Suppose we want to put the Dirac distribution at a different location?

We can express this as

\[
\delta(x - x_0)
\]

(6.66)

What if we want the Dirac distribution to have a greater magnitude? (That is, have an integral greater than 1). Just use a multiplier:

\[
M\delta(x - x_0) \implies \int_{-\infty}^{\infty} M\delta(x - x_0) \, dx = M \int_{-\infty}^{\infty} \delta(x - x_0) \, dx = M
\]  

(6.67)

### 6.3.4 Heaviside functions

Let us suppose that we have a Dirac distribution centered at 0 with magnitude 1. Suppose we perform the following integral:

\[
\int_{-\infty}^{t} \delta(x) \, dx
\]  

(6.68)

What will the value of this integral be for \( t < 0 \)? It will be zero.

What is the value for \( t > 0 \)? It will be 1.

Let's give this thing a name:

\[
u(t) = \int_{-\infty}^{t} \delta(x) \, dx = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}
\]

(6.69)
This is called the **Heaviside function**, also called the “Step function.” From the fundamental theorem of calculus we see that, by definition,

\[ \delta(x) = \frac{du(x)}{dx} \]  

(6.70)

### 6.3.5 Ramp functions

What happens if we integrate the Heaviside function?

For \( x < 0 \) we just have 0, but for \( x > 0 \) we get \( x \). So we have:

\[
\int_{-\infty}^{x} u(\hat{x}) d\hat{x} = \begin{cases} 
0 & x < 0 \\
x & x > 0
\end{cases} = x u(x) = r(x) 
\]

(6.71)

where \( r(x) \) is called the **ramp function**.

We can write the following:

\[
\alpha \delta(x - x_0) = \frac{d}{dx}(\alpha u(x - x_0)) \quad \quad \alpha u(x - x_0) = \frac{d}{dx}(\alpha r(x - x_0))
\]

(6.72)

We can use this to solve beam equations much more easily.
6.3.6 Beam problems with point loads

Previously, we found a nice relationship between distributed load, shear, and moment:

\[
\frac{dV(x)}{dx} = w(x) \quad \frac{dM(x)}{dx} = V(x)
\]

But how does this work for point loads? We can think of a point load as a distributed load that has been concentrated to a single point on a beam, and represent this using a Dirac delta:

**Example 6.6**

Consider the following beam subjected to a combination of distributed loads and point loads.

With the old technology, this would be a difficult problem to solve. Let’s see if we can solve it more easily by representing everything as a single distributed load.

\[
w(x) = f_a\delta(x - L) + f_b\delta(x - 2L) - \frac{w_0}{3L} x
\]

Now, we can integrate this thing easily to obtain the shear:

\[
V(x) = \int w(x)dx = f_a\int \delta(x - L)dx + f_b\int \delta(x - 2L)dx - \frac{w_0}{3L}\int xdx
\]

\[
= f_a u(x - L) + f_b u(x - 2L) - \frac{w_0 x^2}{6L} + C
\]

Solve for \(C\) using the free-end boundary condition:

\[
V(0) = 0 + C = 0 \implies C = 0
\]

Now, integrate again to obtain the moment:

\[
M(x) = \int V(x)dx = f_a\int u(x - L)dx + f_b\int u(x - 2L)dx - \frac{w_0}{6L}\int x^2dx
\]

\[
= f_a(x - L)u(x - L) + f_b(x - 2L)u(x - 2L) - \frac{w_0 x^3}{18L} + D
\]

Using the fact that \(M(x) = 0\) we get \(D = 0\).
Now, we can compute our shear and moment reactions quite easily simply by plugging in $x = 3L$:

$$V(3L) = f_a u(3L - L) + f_b u(3L - 2L) - \frac{w_0(3L)^2}{6L} = f_a + f_b - \frac{3w_0L}{2}$$  \hfill (6.80)

$$M(3L) = f_a (3L - L) u(3L - L) + f_b (3L - 2L) u(3L - 2L) - \frac{w_0}{18L} (3L)^3$$  \hfill (6.81)

$$= 2Lf_a + Lf_b - \frac{3L^2w_0}{2}$$  \hfill (6.82)

Much easier and cleaner than the previous way!

### 6.3.7 Unit doublet

How do we represent point moments in this framework? Suppose we have a point couple moment with magnitude $M_0$ acting at a location $x_0$. Then we would expect that

$$M(x) = M_0 u(x - x_0) \implies V(x) = \frac{dM}{dx} = M_0 \delta(x - x_0)$$  \hfill (6.83)

In general, we don’t want to prescribe a loading in terms of $V(x)$ on our beam; we would rather prescribe it in terms of the loading function $w(x)$. If we use our differential relationship, that indicates that

$$w(x) = \frac{dV}{dx} = M_0 \frac{d}{dx} \delta(x - x_0)$$  \hfill (6.84)

How do we take the derivative of the Dirac delta? This is one of the places where we start to run into trouble: the delta function is not actually a function, and so we can’t exactly take derivatives of it in the usual way. However, if we treat it as a distribution, functional analysis allows us to define the distributional derivative of it, which is called the **unit doublet**, $u_1(x)$.

We can roughly visualize the unit doublet as the limit of the following:

where we let the width of the function go to zero and the magnitude approach infinity. You can see that if we applied a loading $w(x)$ of this form, we would be applying “equal and opposite” forces that are infinite in magnitude and infinitesimally close to each other. In other words, there would be no net force, but there would be a resultant couple moment.

### 6.3.8 “Singularity function” notation

An alternative notation is frequently used that is referred to as “singularity brackets”. They are frequently seen in beam theory, and are identical to the Dirac delta/Heaviside/ramp functions that we are using except for notation.
Singularity functions are written in the following general form:

\[(x - x_0)^n\]  \hspace{1cm} (6.85)

where \(x_0\) is the location of the singularity and \(n\) is the order of the function. A Heaviside is “order 0,” a ramp function is “order 1” and a Dirac delta is “order -1,” that is:

\[(x - x_0)^-2 = u_1(x - x_0) \equiv \text{unit doublet}\] \hspace{1cm} (6.86)

\[(x - x_0)^-1 = \delta(x - x_0) \equiv \text{Dirac delta}\] \hspace{1cm} (6.87)

\[\begin{cases} 
1 & x > x_0 \\
0 & x < x_0
\end{cases} \equiv u(x - x_0) \equiv \text{Heaviside function}\] \hspace{1cm} (6.88)

\[\begin{cases} 
(x - x_0)^1 & x > x_0 \\
0 & x < x_0
\end{cases} \equiv r(x - x_0) \equiv \text{ramp function}\] \hspace{1cm} (6.89)

\[\begin{cases} 
(x - x_0)^2 & x > x_0 \\
0 & x < x_0
\end{cases} \equiv \text{...}\] \hspace{1cm} (6.90)

\[\begin{cases} 
(x - x_0)^n & x > x_0 \\
0 & x < x_0
\end{cases} \equiv \text{Heaviside function}\] \hspace{1cm} (6.91)

You may encounter this notation in mechanics of materials, and feel free to use it on your homework if you like.

### 6.4 Introduction to Euler-Bernoulli beam theory

This section is somewhat outside of the typical statics curriculum; however, it dovetails very nicely with the beam theory that we have discussed. We will go over the derivation of the theory, but you will only be required to use the result for homework problems.

We have introduced the relationship between applied loading, shear force, and moment. It turns out that this relationship can be extended to predict the deflection of a beam under a specified loading. To link the applied loading to the deflection of the beam, we need to introduce stress and strain:

\[\sigma = \frac{f}{A} \Rightarrow [\text{force}] \frac{\text{length}}{[\text{area}]} \quad \varepsilon = \frac{\Delta \ell}{\ell_0} \Rightarrow [\text{length}] \frac{[\text{length}]}{[\text{length}]} = 1\] \hspace{1cm} (6.92)

We can relate these two quantities together using Young’s Modulus \(E\) in the following linear relationship:

\[\sigma = E \varepsilon\] \hspace{1cm} (6.93)

where \(E\) has the same units as \(\sigma\) of pressure, or (in SI units) Pascals. (6.93) is called Hooke’s Law, and you’ll see that it’s similar to the constitutive model that we use for a linear spring.

Let’s consider a beam that is allowed to bend by a very small amount.

We parameterize the deflection by a function \(h(x)\), and we note that the beam always undergoes “pure bending,” that there is no in-plane shear allowed. We will now proceed to relate \(h(x)\) to the bending moment \(M(x)\) that we have seen earlier.
The strain as a function of $z$ is given by

$$\varepsilon(z) = \frac{\Delta x(z) - \Delta x}{\Delta x} = \frac{\Delta x - z \Delta \theta - \Delta x}{\Delta x} = -z \frac{\Delta \theta}{\Delta x} \xrightarrow{\Delta x \to 0} -z \frac{d\theta}{dx} \approx -z \frac{d^2 h}{dx^2}$$

(6.94)

where the last approximation is legitimate only when $h(x)$ is very small. Because we now know $\varepsilon(z)$, we can compute the stress in the beam. If we plot the stress in the beam, we see that it is a linearly decreasing function of $z$.

Now here's the important connection: the stress in this beam creates a net moment, which is identical to the bending moment that we have been using.

Since we have an expression for the stress, we can express it by integrating the moment over the cross-sectional area of the beam:

$$M(x) = -\int_A z \sigma(y, z) dA = E \frac{d^2 h}{dx^2} \int_A z^2 dA = E I \frac{d^2 h}{dx^2}$$

(6.95)

where

$$I = \int_A z^2 dA$$

(6.96)

is a special quantity called the moment of inertia that we will discuss in greater detail in a subsequent section. Now, we can couple our differential relationship between loading, shear, and moment, with the above expression to write:

$$\frac{d^2}{dx^2} \left[ EI \frac{d^2 h}{dx^2} \right] = \frac{d^2}{dx^2} \left[ EI \frac{d\theta}{dx} \right] = \frac{d^2}{dx^2} M(x) = \frac{d}{dx} V(x) = w(x)$$

(6.97)

where $\theta \approx h'(x)$ is the angle of the beam (in radians).

Recall that last time we introduced the Euler-Bernoulli beam bending equation:

$$\frac{d^2}{dx^2} \left[ EI \frac{d^2 h}{dx^2} \right] = \frac{d^2}{dx^2} \left[ EI \frac{d\theta}{dx} \right] = \frac{d^2}{dx^2} M(x) = \frac{d}{dx} V(x) = w(x)$$

(6.98)

where $h(x)$ is the deflection of the beam, $\theta(x)$ is the angle of the beam, $M(x)$ is the bending moment, $V(x)$ is the shear, and $w(x)$ is the loading function. We also introduced the scalar values $E$ (the Young's modulus) and $I$, the moment of inertia about the bending axis.

### 6.4.1 Moment of Inertia

We will talk about the moment of inertia in detail in a subsequent section. For now, we will work with a simple example. Consider a beam that has a simple rectangular cross-section with width $W$ and height $H$. 

The moment of inertia about the x axis is given by computing the following integral:

\[ I = \int_A y^2 \, dz = \int_{-W/2}^{W/2} \int_{-H/2}^{H/2} y^2 \, dy \, dz = \int_{-W/2}^{W/2} \left[ \frac{1}{3} y^3 \right]_{-H/2}^{H/2} \, dz = \int_{-W/2}^{W/2} \frac{H^3}{12} \, dz = WH^3 \frac{1}{12} \] (6.99)

We will compute moments of inertia that are considerably more complex. For right now, we'll just work with simple square beams. In fact, as far as this course is concerned, you'll just be given a moment of inertia, and you can find your answer in terms of it.

### 6.4.2 Boundary conditions

Notice that the beam bending equation is a *fourth order* differential equation for \( h(x) \). This means that we will need *four* boundary conditions to come to our final solution. Let's look at some familiar examples of beam endings to determine what the corresponding boundary conditions are.

**Free end**

Consider a beam that is free to move at the end.

We recall that because the beam is free to move and to rotate, that no shear or moment can be sustained, and hence

\[
\begin{align*}
 h(x_{\text{free end}}) & \neq 0 \\
 \theta(x_{\text{free end}}) & \neq 0 \\
 V(x_{\text{free end}}) & = 0 \\
 M(x_{\text{free end}}) & = 0
\end{align*}
\] (6.100)

**Pivot/roller end**

Consider a pin constrained by a pivot or a roller as shown:

What are the boundary conditions for this end? We know that it is constrained to move but it is free to rotate. Our boundary conditions, therefore, are

\[
\begin{align*}
 h(x_{\text{pinned end}}) & = 0 \\
 \theta(x_{\text{pinned end}}) & \neq 0 \\
 V(x_{\text{pinned end}}) & \neq 0 \\
 M(x_{\text{pinned end}}) & = 0
\end{align*}
\] (6.101)

**Fixed rotation**

Consider a beam that is free to translate but has constrained rotation:
The boundary conditions here are

\[ h(x_{\text{roller}}) \neq 0 \quad \theta(x_{\text{roller}}) = 0 \]
\[ V(x_{\text{roller}}) = 0 \quad M(x_{\text{roller}}) \neq 0 \]  

(6.102)

It is worth noting here that the boundary conditions here are exactly the opposite of that of the pivot/roller end.

**Fixed end**

Finally, consider a beam that is fixed at one end, that we might draw as:

As usual a fixed end constrains all of the degrees of freedom of our system, so we know that

\[ h(x_{\text{fix}}) = 0 \quad \theta(x_{\text{fix}}) = 0 \]
\[ V(x_{\text{fix}}) \neq 0 \quad M(x_{\text{fix}}) \neq 0 \]  

(6.103)

We note that the constraints for the fixed end are opposite to that of the free end – it is, in some sense, the “opposite” of a free end.

Recall that when we talked about constraints, we emphasized the notion that a constrained degree of freedom always corresponds to an applied load and vice-versa. In the above, we note that if we consider the pairs \((h, V)\) and \((\theta, M)\), exactly one of them is always specified – in other words, it is impossible to specify both the deflection and the shear, or the angle and the moment. (As a side note, you’ll see other “pairs” of variables like these in other branches of physics, such as pressure and volume, stress and strain, or voltage and current.)

The other interesting thing to note is that every constraint imposes exactly two boundary conditions. Because every beam must have exactly two ends, and each beam equation requires four boundary conditions, the problem will never be overconstrained.

### 6.4.3 Solution Strategy

1. Write down loading function \(w(x)\)
2. Integrate four times to get shear, moment, angle, and deflection (and don’t forget integration constants)

\[
V(x) = \int w(x) \, dx \quad M(x) = \int V(x) \, dx \quad \theta(x) = \int \frac{M(x)}{EI} \, dx \quad h(x) = \int \theta(x) \, dx
\]  

(6.104)

3. Identify four boundary conditions, and use them to solve for the four integration constants
4. Santity check: units (of course), plotting or substituting values.

We will now demonstrate this with a couple of examples.
Example 6.7

Consider two identical beams subjected to two identical loadings; one is "simply supported" and one is supported with two fixed joints as shown below:

![Beam Diagram](image)

The beam has a Young’s modulus $E$ and a bending moment of inertia $I$. Our goal is to find the deflection of these two beams. Both beams have identical loading functions:

$$w(x) = -f \delta(x - L)$$  \hspace{1cm} (6.105)

This means that we can immediately find $V(x), M(x), \theta(x), h(x)$ for both beams. We know now that they will be the same up to integration constants.

$$w(x) = -f \delta(x - L)$$  \hspace{1cm} (6.106)

$$V(x) = \int w(x - L) \, dx = - \int f \delta(x - L) \, dx = -f u(x - L) + C_1$$  \hspace{1cm} (6.107)

$$M(x) = \int V(x) \, dx = \int -f u(x - L) \, dx = -f u(x - L)(x - L) + C_1 x + C_2$$  \hspace{1cm} (6.108)

$$\theta(x) = \int \frac{M(x)}{EI} \, dx = -\frac{f}{2} u(x - L)(x - L)^2 + \frac{1}{2} C_1 x^2 + C_2 x + C_3$$  \hspace{1cm} (6.109)

$$h(x) = \int \theta(x) \, dx = -\frac{f}{6} u(x - L)(x - L)^3 + \frac{1}{6} C_1 x^3 + \frac{1}{2} C_2 x^2 + C_3 x + C_4$$  \hspace{1cm} (6.110)

Now, our task is to solve for $C_1, C_2, C_3, C_4$. This is an identical process to what we did earlier (with shear and moment diagrams), just with more constants.

Let’s begin with the simply supported beam. We need four boundary conditions. We know that the pins/rollers restrict vertical movement, so we’ve got $h(0) = h(2L) = 0$ – that’s two boundary conditions. We also know that the ends are free to rotate, so there can be no applied moment; so $M(0) = M(2L) = 0$ gives us the remaining two. Let’s plug in:

$$h(0) = \frac{C_4}{EI} = 0 \implies C_4 = 0$$  \hspace{1cm} (6.111)

$$M(0) = -f u(x - L)(x - L) + C_1 x + C_2 = 0 \implies C_2 = 0$$  \hspace{1cm} (6.112)

$$M(2L) = -f L + C_1 (2L) = 0 \implies C_1 = \frac{f}{2}$$  \hspace{1cm} (6.113)

$$h(2L) = -\frac{f}{6} L^3 + \frac{1}{6} C_1 (2L)^3 + C_3 (2L) = 0 \implies C_3 = -\frac{1}{4} f L^2$$  \hspace{1cm} (6.114)

You may be wondering why we didn’t first compute the reactions at the pivot and the roller. After all that’s been our *modus operandi* for almost every other problem we’ve solved. That is certainly a reasonable way of doing it, but it’s not always possible – for instance, the second beam is definitely statically indeterminate.
What is neat is that we actually get the correct boundary conditions by solving this way! Taking our shear equation \( V(x) \) and plugging in the values of \( x \) at the end, we find

\[
V(0) = -\frac{f}{4} u(-L) + C_1 = \frac{f}{2} \quad V(2L) = -\frac{f}{4} u(L) + C_1 = -\frac{f}{2}
\]  

(6.116)

exactly as we expected!

Now, let's consider the second beam. With the old technology, we would be completely stuck because the beam is statically indeterminate. This is because we had relied on our ability to find the shear and moments at the end points to get our boundary conditions. We definitely can't do that here, but what other kinds of boundary conditions can we use? We know that the beam is fixed, so we conclude that \( h(0) = h(2L) = 0 \) and \( \theta(0) = \theta(2L) = 0 \) – that is, the beam can neither shift nor rotate at the fixed joint.

Let's substitute these boundary conditions to solve for our integration constants:

\[
h(0) = \frac{C_4}{EI} = 0 \quad \implies C_4 = 0
\]

(6.117)

\[
\theta(0) = \frac{C_3}{EI} = 0 \quad \implies C_3 = 0
\]

(6.118)

\[
h(2L) = \frac{-\frac{f}{6} (L)^3 + \frac{1}{2} C_1 (2L)^3 + \frac{1}{2} C_2 (2L)^2}{EI} = 0 \quad \implies 8LC_1 + 12C_2 = fL
\]

(6.119)

\[
\theta(2L) = \frac{-\frac{f}{2} (L)^2 + \frac{1}{2} C_1 (2L)^2 + C_2 (2L)}{EI} = 0 \quad \implies 4LC_1 + 4C_2 = fL
\]

(6.120)

We've got a 2x2 system for \( C_1, C_2 \), and we're pretty comfortable with doing that by now. The results are:

\[
C_1 = \frac{4fL - 12fL}{32L - 48L} = \frac{f}{2} \quad C_2 = \frac{8fL^2 - 4fL^2}{32L - 48L} = -\frac{1}{4} \frac{fL}{4}
\]

(6.121)

Now that we've solved for our constants, we know everything we need to know about the beam. In fact, we can actually compute the shear and moment reactions at the end points – that's something we definitely could not do before!

\[
V(0) = -\frac{f}{4} u(0) - \frac{f}{4} L + C_1 = \frac{f}{2} \quad V(2L) = -\frac{f}{4} u(L) + C_1 = -\frac{f}{2} + \frac{f}{2} = -\frac{f}{2}
\]

(6.122)

\[
M(0) = C_2 = -\frac{f}{4} L \quad M(2L) = -\frac{f}{2} L + 2C_1 L + C_2 = -\frac{f}{2} L + fL - \frac{1}{4} fL = \frac{fL}{4}
\]

(6.123)

Substituting the values for the constants, we get the deflection equations \( h(x) \)

\[
h_1(x) = \frac{f}{EI} \left[ -\frac{1}{6} u(x - L)(x - L)^3 + \frac{1}{12} x^3 - \frac{1}{4} L^2 x \right]
\]

(6.124)

\[
h_2(x) = \frac{f}{EI} \left[ -\frac{1}{6} u(x - L)(x - L)^3 + \frac{1}{12} x^3 - \frac{1}{8} L^2 x \right]
\]

(6.125)

What is the maximum deflection of each beam? Assuming it happens in the center (which is pretty valid – although you can also prove this yourself), we have

\[
h_1(L) = \frac{f}{EI} \left[ -\frac{1}{6} u(L - L)(L - L)^3 + \frac{1}{12} L^3 - \frac{1}{4} L^3 \right] = -\frac{fL^3}{6EI}
\]

(6.126)

\[
h_2(L) = \frac{f}{EI} \left[ -\frac{1}{6} u(L - L)(L - L)^3 + \frac{1}{12} L^3 - \frac{1}{8} L^3 \right] = -\frac{fL^3}{24EI}
\]

(6.127)

so we see that

If we substitute value and plot the deflection, we'll see something like the following for \( h(x) \).
Example 6.8

Consider a cantilever beam subjected to a uniform load $w_0$ with Young’s modulus $E$ and moment of inertia $I$:

$$w(x) = -w_0$$ (6.128)

Now, we can run through the integration to get our forms for the shear, moment, angle, and deflection.

$$w(x) = w_0$$ (6.129)

$$V(x) = \int w(x) \, dx = -w_0 x + C_1$$ (6.130)

$$M(x) = \int V(x) \, dx = -\frac{1}{2}w_0 x^2 + C_1 x + C_2$$ (6.131)

$$\theta(x) = \int \frac{M(x)}{EI} \, dx = \frac{1}{EI} \left[ -\frac{1}{6}w_0 x^3 + \frac{1}{2}C_1 x^2 + C_2 \right]$$ (6.132)

$$h(x) = \int \theta(x) \, dx = \frac{1}{EI} \left[ -\frac{1}{24}w_0 x^4 + \frac{1}{6}C_1 x^3 + \frac{1}{2}C_2 x^2 + C_3 x + C_4 \right]$$ (6.133)

What are our boundary conditions? At the fixed end we have $h(0) = 0, \theta(0) = 0$. Popping these in, we get

$$h(0) = \frac{1}{EI} \left[ C_4 \right] = 0 \implies C_4 = 0$$ (6.134)

$$\theta(0) = \frac{1}{EI} \left[ C_3 \right] = 0 \implies C_3 = 0$$ (6.135)

At the free end, we have $V(L) = 0, M(L) = 0$. We could easily use these, but for purposes of example, let’s try doing it a different way. Notice that the beam is statically determinant, meaning that we can find $V(0)$ and $M(0)$ directly. Drawing a free body diagram (and remembering our convention for $V, M$) we have
We're pros at doing this kind of problem now, so let's solve for $M(0)$ and $V(0)$ quickly:

\begin{align*}
\sum f_y &= V(0) - \int_0^L w_0 \, dx = V(0) - w_0 L = 0 \implies V(0) = w_0 L \tag{6.136} \\
\sum M_z &= -M(0) - \int_0^L w_0 x \, dx = -M(0) - \frac{w_0 L^2}{2} = 0 \implies M(0) = -\frac{w_0 L^2}{2} \tag{6.137}
\end{align*}

Now, we can drop these into our equations for $M$ and $V$ to solve for the integration constants:

\begin{align*}
M(0) &= C_2 = -\frac{w_0 L^2}{2} \implies C_2 = -\frac{w_0 L^2}{2} \tag{6.138} \\
V(0) &= C_1 = w_0 L \implies C_1 = w_0 L \tag{6.139}
\end{align*}

Now that we've solved for all of our constants, we can write down our final expression for the deflection of the beam:

$$h(x) = \frac{w_0}{EI} \left[ -\frac{x^4}{24} + \frac{L x^3}{6} - \frac{L^2 x^2}{4} \right]$$ \tag{6.140}

As a sanity check, let's make sure that we recover the other boundary conditions $M(L) = 0$, $V(L) = 0$ with our solution:

\begin{align*}
V(L) &= -w_0 L + w_0 L = 0 \checkmark \tag{6.141} \\
M(L) &= -\frac{1}{2} w_0 L^2 + w_0 L^2 - \frac{1}{2} w_0 L^2 = 0 \checkmark \tag{6.142}
\end{align*}

We're good to go there, so now we can get deflection at the end by evaluating $h(L)$:

$$h(L) = \frac{w_0}{EI} \left[ -\frac{L^4}{24} + \frac{L^4}{6} - \frac{L^4}{4} \right] = -\frac{w_0 L^4}{8EI}$$ \tag{6.143}

Plotting the deflection we have:
6.5 Cables

One more interesting example of internal forces is that of a cable subjected to a distributed load.

We already know how to find the forces and tensions in cables, but now we want to find the shape of the cable under a loading. To do this, we will describe the shape of the cable using a graph $y(x)$: Now, let us find an equilibrium equation by considering a small section of the cable:

Let us do force equilibrium: in the $x$ direction we have:

$$\sum f_x = f_x(x + \Delta x) - f_x(x) = 0 \Rightarrow \frac{f_x(x + \Delta x) - f_x(x)}{\Delta x} = 0 \Rightarrow \frac{df_x(x)}{dx} = 0$$ (6.144)

In the $y$ direction we have

$$\sum f_y = f_y(x + \Delta x) - f_y(x) - w(x)\Delta x = 0 \Rightarrow \frac{f_y(x + \Delta x) - f_y(x)}{\Delta x} = w_0 \Rightarrow \frac{df_y(x)}{dx} = w(x)$$ (6.145)

But we also know something interesting about $f$: we know that

$$f(x) = t(x)n(x)$$ (6.146)

How do we compute the unit vector here? We know that the position vector for the cable is

$$\begin{bmatrix} x \\ y(x) \end{bmatrix}$$ (6.147)

The unit vector is the tangent vector:

$$\frac{d}{dx} \begin{bmatrix} x \\ y(x) \end{bmatrix} = \begin{bmatrix} 1 \\ y'(x) \end{bmatrix}$$ (6.148)
Normalizing, we have
\[
\frac{1}{\sqrt{1 + y'^2(x)}} \left[ \frac{1}{y'(x)} \right]
\]
so our force vector is
\[
f(x) = \frac{t(x)}{\sqrt{1 + y'^2(x)}} \left[ \frac{1}{y'(x)} \right]
\]
(Note: the tension in cables under distributed loads are not constant!) Let us plug this expression into the force balance for \(x\):
\[
\frac{d}{dx} \left( \frac{t(x)}{\sqrt{1 + y'^2(x)}} \right) = 0
\]
This tells us that the force in the \(x\) direction is a constant. We will define
\[
f_x \triangleq \frac{t(x)}{\sqrt{1 + y'^2(x)}} = \text{const}
\]
We can also plug this expression into our force balance for \(y\) to obtain
\[
\frac{d}{dx} \left( \frac{t(x)y'(x)}{\sqrt{1 + y'^2(x)}} \right) = w(x)
\]
We notice that we can substitute \(f_x\) to get
\[
\frac{d}{dx} \left( \frac{t(x)y'(x)}{\sqrt{1 + y'^2(x)}} \right) = f_x \frac{d}{dx} y'(x) = w(x)
\]
Rearranging we have an integral equation for \(y(x)\)
\[
y''(x) = \frac{1}{f_x} w(x) \quad \Rightarrow \quad y(x) = \frac{1}{f_x} \int \left( \int w(x) \, dx \right) \, dx
\]
Let's show how this works with an example:

**Example 6.9**
Consider a suspension bridge where the support cable is loaded with a uniform distributed load \(w(x) = w_0\)

The maximum sag is \(H\) and the length of the segment is \(L\). Find the shape and tension of the cable.
Let us begin by defining a coordinate system and a parameterization \(y(x)\) to describe the cable shape.
Now, we simply solve the differential equation:

\[ y''(x) = \frac{1}{f_x} w(x) = \frac{w_0}{f_x} \implies y(x) = \frac{w_0}{2f_x} x^2 + Ax + B \]  \hspace{1cm} (6.156)

We have three constants to solve for: \( A, B, f_x \). We begin by noting that \( y(0) = 0, y'(0) = 0 \) so

\[ y(0) = B = 0 \quad y'(0) = A = 0 \]  \hspace{1cm} (6.157)

So we have

\[ y(x) = \frac{w_0}{2f_x} x^2 \]  \hspace{1cm} (6.158)

Now, we solve for \( f_x \). We can do this knowing that \( y(L/2) = H \):

\[ y(L/2) = \frac{w_0}{2f_x} \left( \frac{L}{2} \right)^2 = \frac{w_0 L^2}{8f_x} = H \implies f_x = \frac{w_0 L^2}{8H} \]  \hspace{1cm} (6.159)

Substituting, we obtain:

\[ y(x) = \frac{w_0 x^2}{2} \times \frac{8H}{w_0 L^2} = \frac{4Hx^2}{L^2} \]  \hspace{1cm} (6.160)

Finally, we get the tension by the definition of \( f_x \):

\[ t(x) = f_x \sqrt{1 + y''(x)} = \frac{w_0 L^2}{8H} \sqrt{1 + \left( \frac{8Hx}{L^2} \right)^2} = \frac{w_0 L^2}{8H} \sqrt{1 + \frac{64H^2 x^2}{L^4}} \]  \hspace{1cm} (6.161)

### 6.5.1 Point loads

What happens when a cable is subjected not to a continuous load but to point loads? For instance, consider the following cable subjected to two forces \( f_s, f_b \). What shape would you expect the cable to take on?

![Cable with point loads](image)

Fortunately, we have all of the machinery that we need to analyze this problem; all we have to do is use Dirac delta functions to describe our point loads. We'll illustrate this with an example.

**Example 6.10**

Consider a point load with magnitude \( f_s \) acting on the center of a cable with length \( \ell \) as shown:
We can proceed exactly the same way we did before, except now we use a Dirac delta function to describe the point load:

\[ w(x) = f_a \delta(x - W) \]  

(6.162)

Now, we simply integrate, remembering that \( \int \delta(x - x_0) = u(x - x_0) \) and so on:

\[
y(x) = \frac{1}{f_x} \int \left( \int w(x) dx \right) dx = \frac{1}{f_x} \int \left( f_a u(x - W) C_1 \right) dx
\]

\[ = \frac{1}{f_x} (f_a u(x - W)(x - W) + C_1 x + C_2) \]  

(6.163)

(6.164)

Now, we define a coordinate system. There are multiple ways to do it, but let’s let the left pivot correspond to the origin. We can now solve for one of the integration constants:

\[ y(0) = C_2 = 0 \implies C_2 = 0 \]  

(6.165)

We can solve for the other integration constant in a similar way:

\[ y(2W) = \frac{1}{f_x} (f_a W + C_1 2W) \uparrow 0 \implies C_1 = -\frac{f_a}{2} \]  

(6.166)

Our resulting function for the shape of the cable is

\[ y(x) = \frac{f_a}{f_x} (u(x - W)(x - W) - \frac{x}{2}) \]  

(6.167)

(Note that we did indeed recover a linear shape, exactly as we expected.) Now, as with the previous case, we still have to solve for \( f_x \). In general this is done using some type of geometric constraint on the cable. In our case, we know that the total length of the cable must equal \( \ell \), so we can do the following curve length integral:

\[
\int_0^{2W} \sqrt{1 + y''(x)^2} dx = \int_0^W \sqrt{1 + \frac{f_a^2}{4f_x^2}} dx + \int_W^{2W} \sqrt{1 + \frac{f_a^2}{4f_x^2}} dx = 2W \sqrt{1 + \frac{f_a^2}{4f_x^2}} = \ell
\]  

(6.168)

Now, all we have to do is crunch some of the algebra to solve for \( f_x \)

\[ 1 + \frac{f_a^2}{4f_x^2} = \frac{\ell^2}{4W^2} \implies \frac{f_a^2}{4f_x^2} = \frac{\ell^2}{4W^2} - 1 \implies f_x = \sqrt{\frac{f_a^2 W^2}{\ell^2 - 4W^2}} \]  

(6.169)

Do the units work out? Yes: we recover a force value. What happens if \( \ell^2 < 4W^2 \)? We get either an infinite or a complex value—exactly as we expect. The cable must, geometrically, be longer than \( 2W \). Our final result, then, is

\[ y(x) = \sqrt{\frac{\ell^2 - 4W^2}{W^2}} \left( u(x - W)(x - W) - \frac{x}{2} \right) \]  

(6.170)
(Notice that $f_s$ cancelled out, indicating that the solution does not depend on the magnitude of the force. This is what we expect—the shape should be the same for all $f_s$.)

### 6.5.2 Cable subjected to its own weight

We've considered the case of a cable subjected to a distributed load, but another interesting case is a cable subjected to its own weight. There is a subtle difference between the two, in that the amount of the load on an interval of the cable $\Delta x$ is dependent on the slope of the cable at that point.

\[
\Delta s = f(x + \Delta x) - f(x) \\
\Delta y = \int f(x) \, dx \\
\Delta x = \int dx \\
\left[ \int w_0 \sqrt{1 + y'^2(x)} \, dx \right] dx
\]

In other words, if the cable has a weight $w_0$ (force per unit length), then the loading function is

\[
w_0 \, ds = w_0 \sqrt{dx^2 + dy^2} = w_0 \sqrt{1 + y'^2(x)} \, dx
\]

What happens when we plug this into our formula?

\[
y(x) = \frac{1}{f_x} \int \left[ \int w_0 \sqrt{1 + y'^2(x)} \, dx \right] dx
\]

Unfortunately, we cannot solve this directly because the integrand does not depend specifically on $x$, but on $y$ as well. We'll have to take a different approach: let us try differentiating both sides and rearranging a bit:

\[
f_x y'(x) = \int w_0 \sqrt{1 + y'^2(x)} \, dx
\]

\[
f_x y''(x) = w_0 \sqrt{1 + y'^2(x)}
\]

This is an equation that depends on both the first and second derivatives of $y$. Since $y$ is what we're solving for, this means we have to solve an **ordinary differential equation** (ODE). In fact, this is a particularly tricky type of ODE because it is **nonlinear**. Unfortunately (or fortunately?) the method for solving this ODE is beyond the scope of this class, so we'll just accept that the solution is given by the following:

\[
y(x) = \frac{f_x}{w_0} \cosh \left( C_1 + \frac{w_0}{f_x} x \right) + C_2
\]

The function $\cosh$ is the hyperbolic cosine, and is defined by

\[
\cosh(x) = \frac{e^x + e^{-x}}{2}
\]

The hyperbolic cosine is similar to a quadratic function but is a little more rounded out, as we can see by plotting the two:
7 Friction

Previously we have been solving equilibrium equations for a variety of different problems in which the only equations we need are force balance and moment balance (with the occasional spring constitutive equation). In this section, we are going to briefly switch gears to discuss friction. Before beginning, let us make a few notes and disclaimers.

1. Equations of friction are models, whereas equations of equilibrium are laws. In other words, force and moment balance are always true, but equations of friction are merely educated guesses (usually based on experimental observation) and consequently are subject to error.

2. We will use very simple frictional models. More complex models exist, but the purpose of introducing friction at this point is to make you familiar with how friction interacts with a static system.

3. Our current understanding of friction is fairly bad, and understanding and modeling friction is still an active research area, with many recent papers having been published on the subject.

7.1 Frictional forces

Consider the following block of weight $w$ resting on an inclined plane, and its free body diagram:

What are the forces acting on this block?

- The box has weight $w$, so we know that the force of gravity is acting with magnitude $w$ on the box:

$$\mathbf{w} = \begin{bmatrix} 0 \\ -w \end{bmatrix}$$  \hspace{1cm} (7.1)

- What kind of reaction force do we have at the plane? Let's model it as a roller. This means that the force is a normal force, and it acts orthogonally to the surface. So, we know that $f_n = f_n \mathbf{n}$. What is the normal vector here? By geometry we have:

$$\mathbf{n} = \begin{bmatrix} -\sin \theta \\ \cos \theta \\ 0 \end{bmatrix} \implies f_n = \begin{bmatrix} -f_n \sin \theta \\ f_n \cos \theta \\ 0 \end{bmatrix}$$  \hspace{1cm} (7.2)

We just have one unknown (but two equations). Can we solve this system?

$$\sum \mathbf{f} = \begin{bmatrix} -f_n \sin \theta \\ f_n \cos \theta - w \end{bmatrix} \implies f_n = 0 \& f_n = \frac{w}{\cos \theta} \implies \text{contradiction}$$  \hspace{1cm} (7.3)
Apparently we can’t solve it. And it should make sense that we can’t solve it because this block will simply slide down the plane. To balance our number of unknowns with our number of equations, we need another unknown. So, let us introduce a frictional force that acts tangentially to the surface:

- The frictional force.

![Diagram of friction forces](image)

We know the direction of the force so we can write

\[ f_r = f_r \begin{bmatrix} \cos \theta \\ \sin \theta \\ 0 \end{bmatrix} \] (7.4)

(Notice that \( f_n \cdot f_r = f_n f_r \left( -\sin \theta \cos \theta + \sin \theta \cos \theta \right) = 0 \) – friction should always be orthogonal to the normal force.) Now we can solve the system using equilibrium:

\[ \sum f = \begin{bmatrix} f_r \cos \theta - f_n \sin \theta \\ f_r \sin \theta + f_n \cos \theta - w \end{bmatrix} \] (7.5)

Equilibrium in the x direction gives \( f_r = f_r \cos \theta / \sin \theta \); substituting into y-equilibrium equation gives

\[ f_r \sin \theta + f_r \cos^2 \theta / \sin \theta = w \implies f_n \sin^2 \theta + f_r \cos^2 \theta = w \sin \theta \implies f_r = w \sin \theta, f_n = w \cos \theta \] (7.6)

What do we expect to happen as \( \theta \to 90^\circ \)? The block should start sliding; that is, we eventually reach a point where friction fails. How do we model this? We need a constitutive model for friction.

### 7.2 Constitutive Friction Model

To understand how to model friction, we turn to what is observed in experiment. The following figure shows the frictional force as a function of time:

![Friction force vs time](image)

Note that there are two regimes: in the region to the left, the block is static; in the region to the right it is moving. This indicates that there is a range of force that friction can sustain, but after this range, it begins to permit motion. Moreover, we notice that friction can only be sustained when there is also a normal force. Therefore, we introduce the following model:

\[ 0 \leq ||f_r|| \leq \mu_s ||f_n|| \] (7.7)
where \( |f_r| \) is the magnitude of the frictional force (tangent to the surface), \( |f_n| \) is the magnitude of the normal force (normal to the surface), and \( \mu_s \) is an experimentally-determined parameter called the coefficient of static friction.

### Example 7.1

Let us consider the previous problem. Suppose that we measure the coefficient of static friction between the box and the plane to be \( \mu_s \). How high can we tilt the plane before the box starts to move?

The key in this question is “before the box starts to move.” The moment before it begins to move, we know that we have the equality

\[
|f_r| = \mu_s |f_n| \quad (7.8)
\]

We substitute our previous values in to obtain:

\[
w \sin \theta = \mu_s (w \cos \theta) \implies \tan \theta = \mu_s \implies \theta = \tan^{-1}(\mu_s) \quad (7.9)
\]

### 7.3 Solution Strategy

As usual, it is generally helpful to have a “recipe” to follow when solving friction problems. Our solution strategy is similar to what we’ve done before.

1. Identify unknowns: as usual, we will want to find reactions (such as normal forces) and tensions. What about frictional forces – are they known or unknown? Within the context of our simplified model, there are three cases:

   - **No motion**: the frictional force resists the applied force, so \( |f_r| < \mu_s |f_n| \). Then the frictional force \( |f_r| \) is an unknown.
   - **Impending motion**: the frictional force is at its maximum possible value, so \( |f_r| = \mu_s |f_n| \). Then the frictional force does not add an unknown.
   - **Sliding motion**: friction is kinetic, so \( |f_r| = \mu_k |f_n| \). Then the frictional force is a known quantity.

2. Draw FBD for each rigid body. Don’t forget to include frictional forces.

3. Write equations: force and moment equilibrium, and the constitutive equation.

4. Solve

5. Sanity check

6. Substitute numbers

### 7.4 Examples

#### Example 7.2

Consider the following configuration of blocks with weight \( w \) stabilized with a cable and pulleys as shown:
The coefficient of static friction between the contacting surfaces is $\mu_s$. Determine the maximum angle to which the plane can be inclined before the blocks begin to move.

(1) Determine unknowns: we want to find $\theta$ in terms of $\mu_s$, but along the way we will need the tension in the cable $t$, the magnitude of the normal forces $f_{n12}, f_{n23}$, and the magnitudes of the frictional forces $f_{fr12}, f_{fr23}$.

(2) Draw FBD. Before doing this, let us rotate the entire setup by $\theta$ – that will make it easier to find our vectors.

where

$$w_1 = \begin{bmatrix} -w \sin \theta \\ -w \cos \theta \\ 0 \end{bmatrix}, \quad f_{n12} = \begin{bmatrix} 0 \\ f_{n12} \\ 0 \end{bmatrix}, \quad f_{n23} = \begin{bmatrix} 0 \\ f_{n23} \\ 0 \end{bmatrix}, \quad f_{cab1} = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix}, \quad f_{fr12} = \begin{bmatrix} f_{fr12} \\ 0 \\ 0 \end{bmatrix}, \quad f_{fr23} = \begin{bmatrix} f_{fr23} \\ 0 \\ 0 \end{bmatrix}$$

(3) Equations: we have four force equilibrium equations, two for each block:

$$\sum f_{block1} = \begin{bmatrix} -w \sin \theta \\ -w \cos \theta \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ f_{n12} \\ 0 \end{bmatrix} + \begin{bmatrix} f_{fr12} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = 0$$

$$\sum f_{block2} = \begin{bmatrix} -w \sin \theta \\ -w \cos \theta \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ f_{n12} \\ 0 \end{bmatrix} - \begin{bmatrix} f_{fr12} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2t \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} f_{n23} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} f_{fr23} \\ 0 \\ 0 \end{bmatrix} = 0$$

But we have five unknowns. What additional equation do we use? We use the constitutive equation for friction. Since we know that the blocks are about to start sliding, we can use the equality. Additionally,
the cable will force both blocks to start moving at exactly the same time; this means that we have two friction equations:

either \( f_{fr12} = \mu_s f_{n12} \) or \( f_{fr23} = \mu_s f_{n23} \) \hspace{1cm} (7.14)

(Note: what if we removed the cable? Then only one block would start sliding, so we can only use one equation. But we would also have removed an unknown.)

(4) Solve: let us start knocking out unknowns.

**Force-y1:**

\[-w \cos \theta + f_{n12} = 0 \implies f_{n12} = w \cos \theta \] \hspace{1cm} (7.15)

**Force-y2:**

\[-w \cos \theta - f_{n12} + f_{n23} = 0 \implies f_{n23} = 2w \cos \theta \] \hspace{1cm} (7.16)

Using our constitutive equation for friction we get

\[ f_{fr12} = \mu_s w \cos \theta \hspace{1cm} f_{fr23} = 2\mu_s w \cos \theta \] \hspace{1cm} (7.17)

**Force-x1:**

\[-w \sin \theta + \mu_s w \cos \theta + t = 0 \implies t = w(\sin \theta - \mu_s \cos \theta) \] \hspace{1cm} (7.18)

**Force-x2:**

\[-w \sin \theta - f_{fr12} + 2t + f_{fr23} = 0 \] \hspace{1cm} (7.19)

\[-w \sin \theta - \mu_s w \cos \theta + 2w(\sin \theta - \mu_s \cos \theta) + 2\mu_s w \cos \theta = 0 \] \hspace{1cm} (7.20)

\[ \sin \theta - \mu_s \cos \theta = 0 \] \hspace{1cm} (7.21)

\[ \mu_s = \frac{\sin \theta}{\cos \theta} = \tan \theta \] \hspace{1cm} (7.22)

\[ \theta = \tan^{-1}(\mu_s) \] \hspace{1cm} (7.23)

---

**Example 7.3**

Consider the following cylinder of weight \( w \) resting in a corner and subjected to a loading \( f \) with magnitude \( f \).
Between the cylinder and the floor, the coefficient of static friction is \( \mu_s \) and kinetic friction is \( \mu_k \). The wall is smooth. Determine the normal and frictional forces developed at the wall and the floor in terms of \( f \), and the maximum value of \( f \) before the cylinder starts to rotate.

1. Determine unknowns: we are looking for \( f_{n\text{floor}} \), \( f_{k\text{floor}} \), \( f_{n\text{wall}} \).

2. Draw FBD:

![Free Body Diagram]

where

\[
\mathbf{f} = \begin{bmatrix} 0 \\ f \end{bmatrix} \quad \mathbf{w} = \begin{bmatrix} 0 \\ -w \end{bmatrix} \quad \mathbf{f}_{n\text{floor}} = \begin{bmatrix} 0 \\ f_{n\text{floor}} \end{bmatrix} \quad \mathbf{f}_{k\text{floor}} = \begin{bmatrix} f_{k\text{floor}} \\ 0 \end{bmatrix} \quad \mathbf{f}_{n\text{wall}} = \begin{bmatrix} -f_{n\text{wall}} \\ 0 \end{bmatrix}
\]  

(7.24)

3. Write equations: force equilibrium in the x and y directions gives:

\[
\sum \mathbf{f} = \begin{bmatrix} 0 \\ f \end{bmatrix} + \begin{bmatrix} 0 \\ -w \end{bmatrix} + \begin{bmatrix} 0 \\ f_{n\text{floor}} \end{bmatrix} + \begin{bmatrix} f_{k\text{floor}} \\ 0 \end{bmatrix} + \begin{bmatrix} -f_{n\text{wall}} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]  

This is only two equations, but we have three unknowns. We also need to do moment equilibrium.

\[
\sum \mathbf{M} = \begin{bmatrix} 0 \\ 0 \\ -f_r \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ R f_{k\text{floor}} \end{bmatrix} = 0
\]  

(7.26)

Finally, we have the friction inequality:

\[
|f_{k\text{floor}}| \leq \mu_s |f_{n\text{floor}}|
\]  

(7.27)

4. Solve:

   force-y:

\[
f_{n\text{floor}} = w - f
\]  

(7.28)

moment-z:

\[
R f_{n\text{floor}} - rf = 0 \implies f_{n\text{floor}} = \frac{r}{R} f
\]  

(7.29)
force-x:

\[ f_{frfloor} - f_{nwall} = 0 \implies f_{nwall} = \frac{r}{R} f \quad (7.30) \]

To determine the maximum possible load \( f \) that can be applied before the cylinder starts to rotate, use the constitutive relation where the inequality has been replaced with an equality:

\[ f_{frfloor} \leq \mu_s f_{frfloor} \implies \frac{r}{R} f \leq \mu_s (w - f) \implies f \leq \frac{\mu_s w}{r/R + \mu_s} \quad (7.31) \]
8 Principle of Minimum Potential Energy

The previous two sections (center of gravity and moment of inertia) were somewhat of a departure from the classical statics problems that we have been working with. In this section, we will return to dealing with problems of this type. In previous problems we were concerned with finding member forces or reaction forces for a certain configuration. In the problems we work with here, we are interested in finding equilibrium configurations; that is, given a system, find the configuration that causes it to satisfy equilibrium. This generally means that we will be solving for things like angles or lengths rather than forces. Before we introduce energy specifically, it will be helpful for us to define degrees of freedom.

8.1 Degree of freedom

Definition 8.1. The degree of freedom of a system is the number of independent variables needed to define its configuration.

Consider the following three systems:

![Diagram 1](image1)

The state of the pendulum is defined completely by its angle $\theta$. Similarly, the state of the beam assembly is defined by the length $x$, and the spring is determined by its elongation. We say that these are systems with one degree of freedom.

Alternatively, consider the following three systems:

![Diagram 2](image2)

For each case, two variables are required to specify the state of the system. These are systems with two degrees of freedom.

8.2 Potential Energy

Energy is a construction that we introduce to help us solve problems. At its core, it is nothing other than a bookkeeping device. However, energy methods are extremely powerful and help us to solve extremely complicated systems in a very simple way. When considering static systems, we are interested in computing the potential energy. We do this by considering individual components.
8.2.1 Springs

Consider a spring with spring constant $k$ and undeformed length $x_0$.

The spring’s potential energy is given by

$$W_{\text{spring}}(x) = \frac{1}{2} k (x - x_0)^2$$ (8.1)

Note that differentiating gives

$$\frac{dW_{\text{spring}}}{dx} = k (x - x_0) = f$$ (8.2)

the force required to stretch the spring.

8.2.2 Torsion Springs

Consider a coil-shaped spring—this is called a torsion spring.

It twists by an angle $\theta$ when a moment is applied to it. The energy of the torsion spring is

$$W_{\text{coil}}(\theta) = \frac{1}{2} \kappa \theta^2$$ (8.3)

Note that differentiating gives

$$\frac{dW_{\text{coil}}}{d\theta} = \kappa \theta = M$$ (8.4)

the applied moment.

8.2.3 Gravity

Consider a box of mass $m$ in a gravitational field $g$.

$$m$$

arbitrary point
The box’s potential energy is

\[ W_{\text{box}}(x) = mg (x - x_0) \]  

(8.5)

where \( x \) is the distance to an arbitrarily selected fixed point. Note that differentiating gives

\[ \frac{dW_{\text{box}}}{dx} = -mg = w \]  

(8.6)

the weight of the box, or the force exerted on the box by gravity. Also, notice how \( x_0 \) completely disappeared from the force equation.

### 8.3 Energy of applied forces and moments

We need to be precise about one more ingredient before introducing minimum potential energy. Let us consider (i) a force that causes a body to move and (ii) a moment that causes a body to rotate.

What is the work done?

\[ W_{\text{force}} = \int_{u_1}^{u_2} f \cdot du \quad \quad W_{\text{moment}} = \int_{\theta_1}^{\theta_2} M \, d\theta \]  

(8.7)

If the forces and moments are both constant, then the work done is just

\[ W_{\text{force}} = f \cdot \Delta u \quad \quad W_{\text{moment}} = M \, \Delta \theta \]  

(8.8)

and if the force is parallel to the displacement then \( W_{\text{force}} = f \, \Delta u \).

### 8.4 Minimum potential energy

We are now in a position to introduce the principal of minimum potential energy (MPE). MPE is called a variational principle, which means that we will use optimization to solve it.

Consider a system with one degree of freedom, \( x \). Then MPE states that the equilibrium position for the system, \( x^* \), is given by

\[ x = \arg \min_x \Pi(x) \]  

(8.9)

where

\[ \Pi(x) = W_{\text{potential}}(x) - W_{\text{applied}}(x) \]  

(8.10)

“argmin” means that we find the minimizing value for \( x \); \( W_{\text{potential}} \) is the potential energy (e.g. from springs, gravity) and \( W_{\text{applied}} \) is the work done on the system by applied loads.

You may remember from calculus that in order to optimize a function \( f(x) \), we set its derivative equal to zero, then solve for \( x \). We do exactly the same thing here:

\[ \frac{d}{dx} \Pi(x) = 0 \]  

(8.11)

This is called the stationarity condition, and it gives us an equation that we can solve directly for \( x \).
8.5 Solution strategy for one degree of freedom

We can follow this simple recipe for solving problems with the principle of minimum potential energy.

1. Identify the system’s degree of freedom, e.g. $x$.
2. Write down the potential energy in terms of $x$.
3. Write down the work done by external forces in terms of $x$.
4. Write down the total energy function $\Pi(x)$
5. Solve the stationarity condition

$$\frac{d}{dx} \Pi(x) = 0$$

for $x$.

6. Sanity check

7. Substitute numbers

---

**Example 8.1**

Consider a box of weight $w$ attached to a spring with spring constant $k$ and undeformed length $\ell_0$.

![Diagram of a box attached to a spring](image)

Find the equilibrium value of $x$.

We identify that our system has one degree of freedom, $x$. Now, let us construct the potential energy as a function of $x$.

For the spring, we have

$$W_{\text{spring}}(x) = \frac{1}{2} (x - \ell_0)^2$$

(8.13)

The box is in a gravitational field, so we have

$$W_{\text{box}}(x) = -w_0 x$$

(8.14)

Our energy function is

$$W(x) = W_{\text{spring}}(x) + W_{\text{box}} = \frac{1}{2} k (x - \ell_0)^2 - w_0 x$$

(8.15)

(Note that we don't have any applied loads.)

Next step: minimize $W(x)$ with respect to $x$. We do this by setting the derivative of $W(x)$ with respect to $x$ equal to zero:

$$\frac{d}{dx} W(x) = \frac{d}{dx} \left[ \frac{1}{2} k (x - \ell_0)^2 - w_0 x \right] = k(x - \ell_0) - w_0 = 0$$

(8.16)
Solving for $x$ we get

$$x = \frac{w_0}{k} + \ell_0 \quad (8.17)$$

Let us check our answer by using the old technology: force balance. We recall that the force on a spring is equal to $f = k(x - \ell_0)$. With this information, let us do a force balance:

$$\sum f_y = k(x - \ell_0) - w = 0 \implies x = \frac{w}{k} + \ell_0$$

which gives us the same answer.

We can verify that we do indeed get the same answer, but how is this better than doing force balance? We can see the advantage by applying this method to a more complicated problem:

**Example 8.2**

Consider the following linkage subjected to a horizontal force of magnitude $f$. The spring is unstretched at length $L$.

Find the equilibrium distance $x$.

- Our degree of freedom is $x$.
- We only have one potential energy term:
  $$W_{\text{spring}} = \frac{1}{2} k (x - L)^2 \quad (8.19)$$
- We need to compute $\Delta u$, the distance of the joint. When there is no force applied, the $x$-distance from the pivot is $L/2$; with the force applied, it is $x/2$. Therefore:
  $$W_{\text{force}} = \frac{1}{2} f (x - L) \quad (8.20)$$
- Our total energy function is:
  $$\Pi(x) = \frac{1}{2} k (x - L)^2 - \frac{1}{2} f (x - L) \quad (8.21)$$
- Now, differentiate $\Pi$ with respect to $x$ and set it equal to zero:
  $$\frac{d}{dx} \Pi(x) = k (x - L) - \frac{f}{2} = 0$$
  $$\quad (8.22)$$
solving for x gives

\[ x = \frac{f}{2k} + L \] (8.23)

We can also compute the equilibrium angle \( \theta \)

\[ \cos \theta = \frac{x/2}{L} = \frac{f}{4Lk} + \frac{1}{2} \] (8.24)

Alternatively... we can do the entire problem again, but finding our potential energy in terms of \( \theta \):

- Our degree of freedom is \( \theta \)
- The energy of the spring in terms of \( \theta \) is

\[ W_{\text{spring}}(\theta) = \frac{1}{2} k [2L \cos \theta - L] = \frac{1}{2} k L^2 [2 \cos \theta - 1]^2 \] (8.25)

- The work done by the force is

\[ W_f(\theta) = f (L \cos \theta - L \cos \theta_0) \] (8.26)

where \( \theta_0 \) is the angle when no force is applied. (It will turn out to be irrelevant.)

- Our total energy function is

\[ \Pi(\theta) = \frac{1}{2} k L^2 [2 \cos \theta - 1]^2 - f (L \cos \theta - L \cos \theta_0) \] (8.27)

- Differentiating with respect to \( \theta \):

\[ \frac{d}{d\theta} \Pi(\theta) = k L^2 [2 \cos \theta - 1] (-2 \sin \theta) + f L \sin \theta = 0 \] (8.28)

Solving for \( \theta \) gives

\[ \left[ f - 2k L [2 \cos \theta - 1] \right] \sin \theta = 0 \] (8.29)

We have two possible solutions: the first is that \( \sin \theta = 0 \) or \( \theta = 0 \). The second is:

\[ f - 2k L [2 \cos \theta - 1] = 0 \] (8.30)

\[ \cos \theta = \frac{f}{4kL} + \frac{1}{2} \] (8.31)

Notes:

- We arrived at the same answer using different degrees of freedom. This is one of the huge advantages to using minimum potential energy: we can pick whatever degree of freedom is easiest for us, and we'll be guaranteed to get the same answer.
• That said, notice that we got an “extra” answer. Is this answer valid? Yes it is, because the system will actually be balanced in that case. However, it will be **unstable**. Hang on, we’ll talk about this in a second.

• Why didn’t we get two answers the first time? The $\theta = 0$ answer would give us $x = 2L$. But remember that it is impossible for $x$ to be greater than $2L$ in our system. This means that our model “breaks” just as it reaches the equilibrium point, so our calculus is going to be broken there.

Let us consider another example.

**Example 8.3**

Consider the following assembly that supports a box of mass $m$ (in a gravitational field with acceleration $g$).

Find the equilibrium angle $\theta$ as a function of the applied moment $M$.

• Our degree of freedom is $\theta$.

• The height of the box measured from the base is $L \sin \theta$. So the gravitational potential energy is

$$W_{\text{box}}(\theta) = mgL \sin \theta$$  \hspace{1cm} (8.32)

• The work done by the applied moment is

$$W_{\text{moment}}(\theta) = M \theta$$  \hspace{1cm} (8.33)

• Our total energy function is

$$W(\theta) = mgL \sin \theta - M \theta$$  \hspace{1cm} (8.34)

• The stationarity condition is

$$\frac{dW}{d\theta} = mgL \cos \theta - M = 0 \implies \theta = \cos^{-1} \left( \frac{M}{mgL} \right)$$  \hspace{1cm} (8.35)
Example 8.4

Consider the following contraption

All of the beams have length $L$ and mass $m$. The springs have unstretched length $L$ and spring constant $k$. The contraption supports a box of mass $m_{\text{box}}$, and the entire field is in a gravitational field with acceleration $g$. Find the equilibrium configuration of the system.

Using what we've learned, how would we solve this system? We would draw four (at least) free body diagrams, write a bunch of equations, and it would be horrible. Instead, let's solve this using the principle of minimum potential energy.

How many degrees of freedom does this system have? Only one. We can represent any state of this system by specifying the variable $\theta$. Let us compute our energy in terms of $\theta$.

- Springs: the energy of the springs will be
  \[ W_{\text{spr}} = \frac{k}{2} (\ell - \ell_0)^2 \]
  where $\ell$ is the length of the spring and $\ell_0$ is the undeformed length of the spring. What is $\ell$ in terms of $\theta$? $\ell = L \cos \theta$; substitute in
  \[ W_{\text{spr}}(\theta) = \frac{k}{2} (L \cos \theta - L)^2 = \frac{1}{2} k L^2 (\cos \theta - 1)^2 \]

- Beams: the centroids of the beams are in the center, so the effective location of their weights is in the center. Take the reference point to be when $\theta = 0$
Thus the gravitational potential energy of the springs is

\[ W_{\text{beam}} = \frac{1}{2} m g L (\cos \theta - 1) \] (8.38)

- Box: let us take our reference frame to be when \( \theta = 0 \). To get the potential energy we need the change in height of the box:

So we can write our potential energy for the box as:

\[ W_{\text{box}} = m_{\text{box}} g L (\cos \theta - 1) \] (8.39)

Now, we put it all together:

\[
W(\theta) = 3W_{\text{spr}}(\theta) + 4W_{\text{beam}}(\theta) + W_{\text{box}}
\]

\[ = 3 \left[ \frac{1}{2} k L^2 (\cos \theta - 1)^2 \right] + 4 \left[ \frac{1}{2} m g L (\cos \theta - 1) \right] + \left[ m_{\text{box}} g L (\cos \theta - 1) \right] \] (8.41)

\[ = \frac{3}{2} k L^2 (\cos \theta - 1)^2 + 2 m g L (\cos \theta - 1) + m_{\text{box}} g L (\cos \theta - 1) \] (8.42)

To solve for \( \theta \), we set the derivative to zero:

\[
\frac{d}{d\theta} W(\theta) = 3 k L^2 (\cos \theta - 1)(-\sin \theta) + 2 m g L (-\sin \theta) + m_{\text{box}} g L (-\sin \theta) = 0
\]

\[
\sin \theta \left[ 3 k L (\cos \theta - 1) + 2 m g + m_{\text{box}} g \right] = 0
\] (8.44)

Apparently we have two solutions. The first is

\[ \sin \theta = 0 \implies \theta = 0 \] (8.45)

The second is

\[ 3 k L (\cos \theta - 1) + 2 m g + m_{\text{box}} g = 0 \implies \cos \theta = 1 - \frac{2 m g + m_{\text{box}} g}{3 k L} \] (8.46)

\[ \implies \theta = \cos^{-1} \left[ 1 - \frac{2 m g + m_{\text{box}} g}{3 k L} \right] \] (8.47)
Does the solution \( \theta = 0 \) make sense? Technically we could balance the system that way, but it would be **unstable**. We'll explore this further later.

What happens when the argument of the \( \cos^{-1} \) term is smaller than \(-1\)? That is, if
\[
1 - \frac{2m g + m_{\text{box}} g}{3kL} < -1
\]
(8.48)
\[
-2 \frac{m g + m_{\text{box}} g}{3kL} < -2
\]
(8.49)
\[
2 m g + m_{\text{box}} g > 6kL
\]
(8.50)

then there is no solution—the box is too heavy to be supported with this contraption.

Note: credit to Prof. Dennis Kochmann at Caltech for this example

### 8.6 Stability

We have seen in our previous examples how we can frequently get multiple solutions. How do we know which are stable and which are unstable?

Suppose we have a total energy function \( \Pi(x) \) and we know that \( x^* \) solves \( \Pi'(x) = 0 \). Then:
\[
\frac{d^2\Pi}{dx^2}(x^*) < 0 \implies \text{unstable} \quad \frac{d^2\Pi}{dx^2}(x^*) = 0 \implies \text{neutral} \quad \frac{d^2\Pi}{dx^2}(x^*) > 0 \implies \text{stable} \quad (8.51)
\]

#### Example 8.5

Consider a ball of mass \( m \) connected via a

Find the stable and unstable equilibrium configurations.

Let us follow our recipe:

- Our degree of freedom is \( \theta \).
- We have one potential energy term: measuring from the bottommost position, we have
  \[
  W_{\text{ball}}(\theta) = m g \ell (1 - \cos \theta)
  \]
  (8.52)
- No applied forces or moments
- So our energy function is just \( \Pi(\theta) = W_{\text{ball}}(\theta) \).
- Solve:
  \[
  \frac{d}{d\theta} \Pi(\theta) = m g \ell \sin \theta = 0 \implies \theta = n \pi
  \]
  (8.53)
  or, specifically, \( \theta = 0 \) (straight down) and \( \theta = \pi \) (straight up).
Now, let us evaluate the stability of these two points:

\[
\frac{d^2\Pi}{d\theta^2} = m g \ell \cos \theta
\]  

(8.54)

Evaluating gives us

\[
\frac{d^2\Pi}{d\theta^2}(0) = m g \ell \implies \text{stable}
\]

\[
\frac{d^2\Pi}{d\theta^2}(\pi) = -m g \ell \implies \text{unstable}
\]  

(8.55)

Let us use this method to evaluate the stability of our results for the previous example

**Example 8.6**

For our energy function we had

\[
W(\theta) = \frac{3}{2} k L^2 (\cos \theta - 1)^2 + 2 m g L (\cos \theta - 1) + m_{\text{box}} g L (\cos \theta - 1)
\]

(8.56)

\[
\frac{d}{d\theta} W(\theta) = 3 k L (\cos \theta \sin \theta - \sin \theta) - 2 m g \sin \theta - m_{\text{box}} g \sin \theta
\]

(8.57)

Now, let us compute the second derivative:

\[
\frac{d^2}{d\theta^2} W(\theta) = 3 k L (2 \cos \theta \sin^2 \theta - \cos \theta) - 2 m g \cos \theta - m_{\text{box}} g \cos \theta
\]

(8.58)

\[
= 3 k L (2 \cos^2 \theta - 1 - \cos \theta) - 2 m g \cos \theta - m_{\text{box}} g \cos \theta
\]

(8.59)

Now, substitute \( \theta = 0 \) and evaluate:

\[
\frac{d^2}{d\theta^2} W(0) = 3 k L (2(1) - 1 - (1)) - 2 m g (1) - m_{\text{box}} g (1) = -(2 m + m_{\text{box}}) g < 0 \implies \text{unstable}
\]

(8.60)

And we’ll do one more example to cement the process.

**Example 8.7**

Consider a lever spring assembly. Note that the spring is on rollers so it is always vertical. Neglect the mass of the lever. The spring is undeformed when the lever is horizontal. A box of mass \( m \) is attached as shown.

Compute the stable and unstable equilibrium angles \( \theta \).

- Our degree of freedom is \( \theta \).
- We have two potential energy terms. For the spring we have

\[
W_{\text{spring}} = \frac{1}{2} k (\Delta u_{\text{spring}})^2 = \frac{1}{2} k (L \sin \theta)^2 = \frac{L^2 k}{2} \sin^2 \theta
\]

(8.61)
For the box we have
\[ W_{\text{box}} = -mg \Delta u_{\text{box}} = -mg (3L \sin \theta) = -3Lm g \sin \theta \] (8.62)

- There are no other applied loads.
- The total energy function is
\[ \Pi(\theta) = \frac{L^2 k}{2} \sin^2 \theta - 3Lm g \sin \theta \] (8.63)
- Now we solve the stationarity condition:
\[ \frac{d\Pi}{d\theta} = (L^2 k \sin \theta)(\cos \theta) - 3Lm g \cos \theta = \cos \theta [L^2 k \sin \theta - 3Lm g] = 0 \] (8.64)

One solution is \( \cos \theta = 0 \), implying that \( \theta = n\pi + \frac{\pi}{2} \). We obtain the other solution by solving
\[ Lk \sin \theta - 3m g = 0 \] (8.65)
\[ \sin \theta = \frac{3m g}{Lk} \] (8.66)
\[ \theta = \sin^{-1} \left( \frac{3m g}{Lk} \right) \] if \( 3m g \leq Lk \) (8.67)

We have three types of solution:
\[ \theta_1 = \frac{\pi}{2} \quad \theta_2 = \frac{3\pi}{2} \quad \theta_3 = \sin^{-1} \left( \frac{3m g}{Lk} \right) \] (8.68)

Let’s analyze the stability of these solutions. First, we need to compute the second derivative of \( \Pi \):
\[ \frac{d^2\Pi}{d\theta^2} = \frac{d}{d\theta} \left[ L^2 k \sin \theta \cos \theta - 3Lm g \cos \theta \right] = L^2 k (\cos^2 \theta - \sin^2 \theta) + 3Lm g \sin \theta \] (8.69)

Now, substitute our solutions:
\[ \Pi''(\pi/2) = L^2 k(0 - 1) + 3Lm g(1) = -L^2 k + 3Lm g \implies \begin{cases} 3m g > Lk & \text{stable} \\ 3m g < Lk & \text{else} \end{cases} \] (8.70)
\[ \Pi''(3\pi/2) = L^2 k(0 - (-1)) + 3Lm g(-1) = L^2 k - 3Lm g \implies \begin{cases} 3m g < Lk & \text{stable} \\ 3m g > Lk & \text{else} \end{cases} \] (8.71)
\[ \Pi''(\theta_3) = L^2 k(1 - 2 \sin^2 \theta) + 3Lm g \sin \theta = L^2 k \left( 1 - 2 \left( \frac{3m g}{Lk} \right)^2 \right) + 3Lm g \left( \frac{3m g}{Lk} \right) \] (8.72)
\[ = L^2 k - \frac{18m^2 g^2}{k} + \frac{9m^2 g^2}{k} = L^2 k - \frac{9m^2 g^2}{k} \implies \begin{cases} 3m g < Lk & \text{stable} \\ 3m g > Lk & \text{else} \end{cases} \] (8.73)

8.7 Multiple degrees of freedom

Up until now, we have considered only systems with a single degree of freedom. However, we can extend this principle to systems with two or more degrees of freedom.

Consider a multi-degree of freedom system defined by variables \( x_1, x_2, \ldots \). Then the energy function for the system can be written
\[ \Pi(x_1, x_2, \ldots) \] (8.74)
and the stationarity conditions are
\[
\begin{align*}
\frac{\partial \Pi}{\partial x_1} &= 0 \\
\frac{\partial \Pi}{\partial x_2} &= 0 \\
\frac{\partial \Pi}{\partial x_3} &= 0 \\
&\ldots
\end{align*}
\] (8.75)

Note that for each variable in the system we are guaranteed to obtain an equation.

**Example 8.8**

Consider the following spring assembly:

![Spring Assembly Diagram]

The system is defined by the *nodal displacements*, such that when \( u_1 = u_2 = u_3 = u_4 \) the springs are undeformed. A force \( f \) is also applied to the system. Find the nodal displacements.

- Our degrees of freedom are \( u_1, u_2, u_3, u_4 \).
- We will have a potential energy term contribution from each spring. Beginning with the first one: suppose that the undeformed length of the spring is \( \ell \). Then

  \[
  W_{spr1} = \frac{1}{2} k_1 (\ell + u_1 - \ell)^2 = \frac{1}{2} k_1 u_1^2
  \] (8.76)

  Moving to the second one: suppose that the undeformed length is \( \ell \). What is the deformed length of the spring? It would be \( \ell - u_1 + u_2 \), so we have

  \[
  W_{spr2} = \frac{1}{2} k_2 ((\ell - u_1 + u_2) - \ell)^2 = \frac{1}{2} k_2 (u_2 - u_1)^2
  \] (8.77)

  By analogy, we then have

  \[
  W_{spr3} = \frac{1}{2} k_3 (u_3 - u_2)^2 \\
  W_{spr4} = \frac{1}{2} k_4 (u_4 - u_3)^2
  \] (8.78)

- The energy from our applied force is just \( W_f = f u_4 \).
- The total energy function, then, is

  \[
  \Pi(u_1, u_2, u_3, u_4) = \frac{1}{2} k_1 u_1^2 + \frac{1}{2} k_2 (u_2 - u_1)^2 + \frac{1}{2} k_3 (u_3 - u_2)^2 + \frac{1}{2} k_4 (u_4 - u_3)^2 - f u_4
  \] (8.79)

- Now, we compute our stationarity conditions:

  \[
  \begin{align*}
  \frac{\partial \Pi}{\partial u_1} &= k_1 u_1 - k_2 (u_2 - u_1) = (k_1 + k_2) u_1 - k_2 u_2 = 0 \\
  \frac{\partial \Pi}{\partial u_2} &= k_2 (u_2 - u_1) + k_3 (u_3 - u_2) = -k_2 u_1 + (k_2 + k_3) u_2 - k_3 u_3 = 0 \\
  \frac{\partial \Pi}{\partial u_3} &= k_3 (u_3 - u_2) - k_4 (u_4 - u_3) = -k_3 u_2 + (k_3 + k_4) u_3 - k_4 u_4 = 0 \\
  \frac{\partial \Pi}{\partial u_4} &= k_4 (u_4 - u_3) - f = -k_4 u_3 + k_4 u_4 - f = 0
  \end{align*}
  \] (8.80 - 8.83)
Writing our stationarity condition in matrix form, we get

$$
\begin{bmatrix}
  k_1 + k_2 & -k_2 & 0 & 0 \\
  -k_2 & k_2 + k_3 & -k_3 & 0 \\
  0 & -k_3 & k_3 + k_4 & -k_4 \\
  0 & 0 & -k_4 & k_4
\end{bmatrix}
\begin{bmatrix}
  u_1 \\
  u_2 \\
  u_3 \\
  u_4
\end{bmatrix}
= 
\begin{bmatrix}
  0 \\
  0 \\
  0 \\
  f
\end{bmatrix}
$$

(8.84)

Let's note a few things:

- The vector of displacements $u$ is called the vector of nodal displacements.
- The other vector is called the vector of forces. Notice that the first three components are zero—why? They are zero because we did not apply loads at the other three nodes.
- The matrix that we constructed is called the stiffness matrix. Notice how it converts displacements to forces.
- We won’t prove it now, but you can show that all of the eigenvalues of the stiffness matrix are positive. This means that the stiffness matrix has the property of being positive definite, and it means that every solution will be stable.
- The concepts that we have learned here will carry over when you learn about the finite element method.

### 8.8 Virtual work

Consider the following truss.

Let us consider node $a$ in particular, and draw a free body diagram. Like usual, we draw our forces in terms of tensions and unit vectors. In general there is no motion of the node, so no work is done. But what if the node did move?

Suppose that node $a$ is displaced by an arbitrarily small vector

$$
u = \begin{bmatrix} u_x \\ u_y \end{bmatrix}$$

(8.85)

Now, let us try using the principle of minimum potential energy. We have two degrees of freedom at this node: $u_x, u_y$. We don’t have any potential energy terms, but we do have terms from the applied loading:

$$
W(u_x, u_y) = f_c \cdot \mathbf{u} + f_d \cdot \mathbf{u} + f_b \cdot \mathbf{u}
$$

$$
= t_c n_{cx} u_x + t_c n_{cy} u_y + t_d n_{dx} u_x + t_d n_{dy} u_y + t_b n_{bx} u_x + t_b n_{by} u_y
$$

(8.86)
Now let us apply our stationarity condition:

\[
\frac{\partial W}{\partial u_x} = t_c n_{cx} + t_d n_{dx} + t_b n_{bx} = 0 \tag{8.88}
\]

\[
\frac{\partial W}{\partial u_y} = t_c n_{cy} + t_d n_{dy} + t_b n_{by} = 0 \tag{8.89}
\]
9 Center of Gravity

At this point in the course, we make a fairly sharp departure from what we have done before. Previous work has focused primarily on using equilibrium equations to solve for forces at various points. A tacit assumption was made that the shape of the bodies that we considered was somewhat irrelevant; this is valid as long as there are no body forces considered.

Now, we will begin to look at the balance laws for objects with complex shape under the influence of "body forces" such as gravity. To do this, we need to introduce a number of mathematical tools. This material is generally covered in a third semester Calculus course, but may not yet be familiar to students in this course Therefore, we will introduce and review the relevant concepts here.

9.1 Review of multivariable integration

We are familiar with the integration of a function over 1D regions. For instance, suppose we have a function $f(x)$ where $f(x)$ could be, for instance, a weight per unit length. The integral of this function over the line segment $[a, b]$ is

$$\int_a^b f(x) \, dx.$$  \hfill (9.1)

Now, we will discuss how to integrate functions in 2 and 3 dimensions over 2- and 3-dimensional objects.

9.1.1 Area integrals

Suppose we have a function of two variables $f(x, y)$. This function could be, for instance, a force per unit area. What is the integral of this function over the following region?

![Diagram of a rectangular region](image)

The integral is given by the following:

$$\int_0^b \int_0^a f(x, y) \, dx \, dy = \int_0^a \int_0^b f(x, y) \, dy \, dx \equiv \int_\Omega f(x, y) \, dA$$  \hfill (9.2)

where we say that $\Omega = [0, a] \times [0, b]$ is the domain of integration and $dA$ is the differential area element. We also note that when we write out the integral explicitly, we can freely exchange the $x$ and $y$ components.

Example 9.1

Compute the integral of

$$f(x, y) = \sin(\pi x) \sin(\pi y)$$  \hfill (9.3)

over the region $0 \leq x \leq 1, 0 \leq y \leq 1.$
Plotting in the region gives this:

and we want to find what is equivalently the volume under this surface. We simply write out the integral:

\[
\int_{\Omega} f(x, y) \, dA = \int_{0}^{1} \int_{0}^{1} \sin(\pi x) \sin(\pi y) \, dy \, dx = \int_{0}^{1} \sin(\pi x) \left( \int_{0}^{1} \sin(\pi y) \, dy \right) \, dx
\]

(9.4)

\[
= \int_{0}^{1} \sin(\pi x) \left( -\frac{1}{\pi} \cos(\pi y) \right)_{0}^{1} \, dx = \int_{0}^{1} \sin(\pi x) \left( -\frac{1}{\pi} (-1 - 1) \right) \, dx
\]

(9.5)

\[
= \frac{2}{\pi} \int_{0}^{1} \sin(\pi x) \, dx = \frac{4}{\pi^2}
\]

(9.6)

This is straightforward enough as long as we are integrating over rectangular regions. But what about if the region is not rectangular?

How can we integrate \( f(x, y) \) over this shape? We start out by noticing that \( x \) ranges from \( a \) to \( b \), so our \( x \) integral can be over that region. Let us suppose that we have graphs, \( f_1(x) \) and \( f_2(x) \), for the top and bottom of the potato. Then we know that for each point \( x \), \( y \) varies from \( f_1(x) \) to \( f_2(x) \). Thus, we can write our integral as:

\[
\int_{\Omega} f(x, y) \, dA = \int_{a}^{b} \int_{f_1(x)}^{f_2(x)} f(x, y) \, dy \, dx
\]

(9.7)

Note that we can no longer exchange the integrals because the interior integral's bounds depend on the exterior integral's area. (Suppose we really want to integrate \( x \) first? We can, but we have to find a different way to describe the bounds.)
Example 9.2

Integrate \( f(x, y) = e^{x+y} \) over the triangular region below:

We parameterize the shape by identifying the bounds \( 0 \leq x \leq 1 \) and \( 0 \leq y \leq 1-x \). Thus we compute our integral as follows:

\[
\int_{\Omega} f(x, y) \, dA = \int_0^1 \int_0^{1-x} e^{x+y} \, dy \, dx = \int_0^1 \left( e^{x+y} \right)^{1-x} \, dx = \int_0^1 \left( e^{x+y} \right)^{1-x} \, dx \\
= \int_0^1 (e^{x+1-x} - e^{x+0}) \, dx = \int_0^1 (e^x - e^1) \, dx = e^1 \left. -e^x \right|_0^1 = e - e^1 + e^0 = 1
\]

Note: how can we compute the area of a region of arbitrary shape? We get the area simply by doing the integral with \( f(x, y) = 1 \):

\[
\text{area}(\Omega) = \int_{\Omega} dA
\]

For instance, in the above example:

\[
\int_0^1 \int_0^{1-x} dy \, dx = \int_0^1 (1-x) \, dx = \left. x - \frac{1}{2} x^2 \right|_0^1 = 1 - \frac{1}{2} = \frac{1}{2}
\]

9.1.2 Volume integrals

We do volume integrals in exactly the same way that we do area integrals:

\[
\int_{\Omega} f(x, y, z) \, dV = \int_a^b \int_{y_1(x)}^{y_2(x,y)} \int_{z_1(x,y)}^{z_2(x,y)} f(x, y, z) \, dz \, dy \, dx
\]

And, similarly, the volume of a region is just computed as:

\[
\text{volume}(\Omega) = \int_{\Omega} dV
\]

9.1.3 Polar coordinates

Frequently, it is much easier to work in terms of cylindrical coordinates instead of rectangular coordinates (see first lecture). That is, we write \( f(r, \theta) \) instead of \( f(x, y) \), where

\[
r = \sqrt{x^2 + y^2} \quad \theta = \tan^{-1} \left( \frac{y}{x} \right) \\
x = r \cos \theta \quad y = r \sin \theta
\]
When we compute 2D integrals we have $da = dx \, dy$, so that
\[
\int_{\Omega} f(x, y) \, da = \int_{\Omega} f(x, y) \, dx \, dy
\]  
(9.17)

What is $da$ in cylindrical coordinates?

We consider a small differential wedge as a portion of our integration body. What is the length? It is the differential element $dr$. What is the width? We are inclined to say $d\theta$. But remember that $d\theta$ is unitless which means that $da$ would be in units of meters, which is incorrect.

Remember: what is the definition of $\theta$ when $\theta$ is in radians?

\[
\theta = \frac{[\text{length of arc subtended by the angle}]}{[\text{length of the radius}]} 
\]  
(9.18)

This means that the width of our wedge is the differential element $d\theta$ multiplied by the radius. Therefore we have $da = (dr)(r \, d\theta)$, so our integral is
\[
\int_{\Omega} f(r, \theta) \, da = \int_{\Omega} f(r, \theta) \, r \, dr \, d\theta 
\]  
(9.19)

**Example 9.3**

Compute the area of a circle of radius $R$.
Using rectilinear coordinates, we can set up our integral this way:

\[
y = \sqrt{R^2 - x^2}
\]
\[
x = -R \quad x = R
\]
\[
y = -\sqrt{R^2 - x^2}
\]

where the area is
\[
A = \int_{\Omega} dA = \int_{-R}^{R} \int_{-\sqrt{R^2 - x^2}}^{\sqrt{R^2 - x^2}} dy \, dx 
\]  
(9.20)
which would be very difficult to compute. However, we can alternatively write this integral using polar coordinates:

\[
A = \int_0^{2\pi} \int_0^R r \, dr \, d\theta = \int_0^{2\pi} \frac{1}{2} r^2 \bigg|_0^R \, d\theta = \int_0^{2\pi} \frac{1}{2} R^2 \, d\theta = \frac{R^2}{2} (2\pi) = \pi R^2
\]

which we know to be the correct answer.

### 9.1.4 Cylindrical coordinates

We extend polar coordinates to 3D by adding a \( z \) component. In this case \( z \) is exactly the same as \( z \) in rectilinear coordinates.

So for volume integrals, we have

\[
\int_{\Omega} f(r, \theta, z) \, dV = \int_{\Omega} f(r, \theta, z) \, r \, dr \, d\theta \, dz
\]

(Note: another type of coordinate system used in 3D is spherical coordinates. Spherical coordinates are very useful, but we will stick to problems that can be handled by the coordinate systems mentioned up to this point.)

### 9.1.5 Line integrals in 2D and 3D

We are familiar with integrating along lines, as mentioned earlier. But what about integrating along curves? Let us consider a curve that is parameterized as follows:

\[
r(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} \quad a \leq t \leq b
\]

(9.23)

Additionally, let us suppose that we have a function defined along the curve: we will call it \( f(t) \). (\( f(t) \) could be, for instance, a weight per unit length.) What is the integral of \( f(t) \) along the curve? We define a differential element of line length, and we will call it \( d\ell \). Then the integral is:

\[
\int_{\Omega} f(t) \, d\ell
\]

(9.24)
Previously, we related $da$ or $dv$ to $dx, dy, dz$. How can we relate $d\ell$ to known variables? To do this, we use the Pythagorean theorem: what is $d\ell$ in terms of $dx, dy, dz$?

$$d\ell = \sqrt{dx^2 + dy^2 + dz^2} \quad (9.25)$$

But $x, y, z$ are all functions of $t$. So then we can write

$$d\ell = \sqrt{\left(\frac{dx}{dt}\right)^2 dt^2 + \left(\frac{dy}{dt}\right)^2 dt^2 + \left(\frac{dz}{dt}\right)^2 dt^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt \quad (9.26)$$

So our contour integral is

$$\int_a^b f(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt \quad (9.27)$$

What if we want to find the length of the contour? Similarly to before, we just set $f(t) = 1$.

### Example 9.4

Compute the length of the following contour in 2D:

$$r(t) = \begin{bmatrix} R \cos(t) \\ R \sin(t) \end{bmatrix} \quad 0 \leq t \leq 2\pi \quad (9.28)$$

Apparently, this is nothing other than the equation for a circle of radius $R$. We already know that the length should be $2\pi R$, but let’s prove this using contour integration.

$$\int_\Omega d\ell = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \quad (9.29)$$

$$= \int_0^{2\pi} \sqrt{(-R \sin(t))^2 + (R \cos(t))^2} dt \quad (9.30)$$

$$= \int_0^{2\pi} R \sqrt{\sin^2(t) + \cos^2(t)} dt \quad (9.31)$$

$$= \int_0^{2\pi} R dt \quad (9.32)$$

$$= 2\pi R \quad (9.33)$$

exactly as we expected!

### 9.2 2D surface integrals in 3D

Suppose we have a surface described by $z = f(x, y)$

We want to find the differential surface element $da$ for our surface in terms of $f, x, y$ that will allow us to integrate in terms of $dx, dy$. To do that, let us consider the tangent vectors to the projection of the differential element:
The tangent vectors are given by differentiation the position vector $r$ with respect to $x$ and $y$

$$
\begin{align*}
\mathbf{t}_x &= \frac{d}{dx} \begin{bmatrix} x \\ y \\ f(x, y) \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} f(x, y) \\ 0 \end{bmatrix} \quad \Rightarrow \quad dt_x = \begin{bmatrix} 1 \\ 0 \end{bmatrix} dx \\
\mathbf{t}_y &= \frac{d}{dy} \begin{bmatrix} x \\ y \\ f(x, y) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \Rightarrow \quad dt_y = \begin{bmatrix} 0 \\ 1 \end{bmatrix} dy
\end{align*}
$$

(9.34) (9.35)

(where $f_x = \frac{\partial}{\partial x} f$, etc.) We can then find a differential area vector by taking the cross product:

$$
\mathbf{d}a = \mathbf{t}_x \times \mathbf{t}_y = \begin{bmatrix} 1 \\ f_x \\ f_y \end{bmatrix} \quad \Rightarrow \quad ||\mathbf{d}a|| = \mathbf{d}a = \sqrt{1 + f_x^2 + f_y^2} \, dx \, dy
$$

(9.36)

As a result, we can write our integral of a function $g$ over the surface as

$$
\int_{\Omega} g(x, y) \sqrt{1 + f_x^2 + f_y^2} \, dx \, dy
$$

(9.37)

Let's try this out with an example:

**Example 9.5**

Set the integral up for the area of the surface defined by

$$
f(x, y) = \cos(\pi x) \sin(\pi y) \quad -\frac{1}{2} \leq x \leq \frac{1}{2} \quad -\frac{1}{2} \leq y \leq \frac{1}{2}
$$

(9.38)

To compute our differential area element, we compute the derivatives

$$
f_x = -\pi \sin(\pi x) \sin(\pi y) \quad f_y = \pi \cos(\pi x) \cos(\pi y)
$$

(9.39)

So our integral is given by

$$
\int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \sqrt{1 + \pi^2 \sin^2(\pi x) \sin^2(\pi y) + \pi^2 \cos^2(\pi x) \cos^2(\pi y)} \, dx \, dy
$$

(9.40)

It turns out that this integral is a bit tedious to compute, so we won't do it here. In practice, you can evaluate an integral like this using a computer algebra system, or by evaluating it numerically.
In Section 3.6 we introduced the notion of a distributed load. In particular, we were interested in finding the effective force and location of the distributed load. In this section we will look at objects with appreciable weight and complex geometry. We will compute the total force of gravity acting on the object (effective force) and the object’s center of gravity (effective location).

### 9.3 Center of gravity of a volume

Consider an arbitrary body as shown in the following figure.

Using our integration tools, how do we express the volume of the body?

\[
\int_{\Omega} dv
\]

(9.41)

Let us suppose that the body has variable density \( \rho(x) \). What is the mass of the body?

\[
\int_{\Omega} dm(x) = \int_{\Omega} \rho(x) dv
\]

(9.42)

where \( dm(x) = \rho(x) dV \) is the differential mass element. What is the total weight of the body, assuming we are in a gravitational field with acceleration \( g \)?

\[
\int_{\Omega} dw(x) = \int_{\Omega} \rho(x) g dv
\]

(9.43)

Let us be a bit more precise. What is the direction of the gravitational force? Negative in the z direction, in general. Let us represent this by representing the differential force element as a vector:

\[
dw(x) = \begin{bmatrix} 0 \\ 0 \\ \rho(x) g \end{bmatrix} dv
\]

(9.44)

where \( dw(x) \) is the differential element of a body force on the object. How do we integrate this? It’s actually super easy, we just integrate each individual component. So then we have

\[
w = \int_{\Omega} dw(x)
\]

(9.45)

which is our equation for the total force on the object. In our special case of vertical gravity, we have

\[
w = \begin{bmatrix} 0 \\ 0 \\ \int_{\Omega} \rho(x) g dv \end{bmatrix}
\]

(9.46)
Now, let us compute the moment of the body force about, let’s say, the origin. How do we do this? We introduce our distance vector \( r(x) \). If we are choosing the origin as our reference point, we just have that \( r(x) = x \). We compute the total moment vector by integrating the differential moment over \( \Omega \):

\[
\mathbf{M} = \int_{\Omega} d\mathbf{m} = \int_{\Omega} \mathbf{x} \times d\mathbf{w}(x)
\]

Let us suppose that \( d\mathbf{w} \) acts in the z direction only, that is

\[
d\mathbf{w}(x) = \begin{bmatrix} 0 \\ 0 \\ \rho(x) g \end{bmatrix} dv
\]

Then the differential moment is

\[
d\mathbf{m}(x) = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ 0 & 0 & \rho(x) g \end{bmatrix} dv
\]

so the total moment is

\[
\mathbf{M} = \begin{bmatrix} \int_{\Omega} y w(x) dv \\ -\int_{\Omega} x w(x) dv \\ 0 \end{bmatrix}
\]

Let us try to find an effective force (and location) for the weight of the thing.

In this case we have, not surprisingly,

\[
\mathbf{w}_{\text{eff}} = \mathbf{w}
\]

The effective moment just gives us

\[
\mathbf{M}_{\text{eff}} = \mathbf{r}_{\text{eff}} \times \mathbf{w}_{\text{eff}} = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \bar{x} & \bar{y} & \bar{z} \\ \int_{\Omega} \rho(x) g dv & \int_{\Omega} \rho(x) g dv & \int_{\Omega} \rho(x) g dv \end{bmatrix} = \begin{bmatrix} \bar{y} \int_{\Omega} \rho(x) g dv \\ -\bar{x} \int_{\Omega} \rho(x) g dv \\ 0 \end{bmatrix}
\]

This allows us to solve for \( \bar{x}, \bar{y}, \bar{z} \). In general we will have that \( w(x) = \rho(x) g \), although there are some other cases where there are different types of body forces.

\[
\bar{x} = \frac{\int_{\Omega} x \rho(x) g dv}{\int_{\Omega} \rho(x) g dv} \quad \bar{y} = \frac{\int_{\Omega} y \rho(x) g dv}{\int_{\Omega} \rho(x) g dv} \quad \bar{z} = \frac{\int_{\Omega} z \rho(x) g dv}{\int_{\Omega} \rho(x) g dv}
\]

Where we identify the relationship for \( \bar{z} \) by analogy. (In other words, suppose gravity acted in the x direction instead—we would get the same thing but for \( \bar{y}, \bar{z} \). We could prove this a little more rigorously, but it’s not that interesting and doesn’t provide much useful insight.)
Example 9.6

Consider the following cone-shaped object with uniform density $\rho_0$ in a gravitational field with acceleration $g$:

![Diagram of a cone-shaped object](image)

Compute the effective weight (total weight) and the effective location of the weight (centroid).

We get the effective weight simply by integrating over the volume:

$$w_z = \int_{\Omega} \rho_0 g \, dV = \rho_0 g \int_{0}^{2\pi} \int_{0}^{R} \int_{0}^{R-r} r \, dz \, dr \, d\theta$$

(9.54a)

$$= \rho_0 g \int_{0}^{2\pi} \int_{0}^{R} (Rr - r^2) \, dr \, d\theta$$

(9.54b)

$$= \rho_0 g \int_{0}^{2\pi} \int_{0}^{R} \left( \frac{R^3}{2} - \frac{R^3}{3} \right) \, dr \, d\theta$$

(9.54c)

$$= \frac{\rho_0 g R^3 \pi}{6}$$

(9.54d)

$$= \frac{\rho_0 g R^3 \pi}{3}$$

(9.54e)

Now, let us compute the centroid coordinates beginning with $\bar{x}$:

$$\int_{\Omega} x \rho_0 g \, dV = \int_{0}^{2\pi} \int_{0}^{R} \int_{0}^{R-r} r \cos \theta \, r \, dz \, dr \, d\theta$$

(9.55a)

$$= \rho_0 g \int_{0}^{2\pi} \int_{0}^{R} r^2 \cos \theta z \bigg|_{z=0}^{z=R-r} \, dr \, d\theta$$

(9.55b)

$$= \rho_0 g \int_{0}^{2\pi} \int_{0}^{R} (Rr^2 - r^3) \cos \theta \, dr \, d\theta$$

(9.55c)

$$= \rho_0 g \int_{0}^{2\pi} \frac{1}{3} Rr^3 \left|_{r=0}^{r=R} \right. - \frac{1}{4} r^4 \left. \int_{0}^{R} \cos \theta \, d\theta \right.$$  

(9.55d)

$$= \frac{R^4}{12} \rho_0 g \int_{0}^{2\pi} \cos \theta \, d\theta$$

(9.55e)

$$= \frac{R^4}{12} \rho_0 g (\sin(0) - \sin(2\pi)) = 0$$

(9.55f)

This tells us that $\bar{x} = 0$. (Note that we probably could have guessed this just by noticing the symmetry of the
problem.) We can use a similar procedure to show that \( \bar{y} = 0 \).

\[
\int_{\Omega} z \rho_0 g \, dV = \rho_0 g \int_0^{2\pi} \int_0^R \int_{R-r}^{R-r} r \, z \, dz \, d\theta = \rho_0 g \int_0^{2\pi} \int_0^R \left[ \frac{1}{2} z^2 \right]_{R-r}^0 r \, dr \, d\theta
\]  
\[
= \frac{1}{2} \rho_0 g \int_0^{2\pi} \int_0^R (R^2 - 2Rr + r^2) r \, dr \, d\theta = \frac{1}{2} \rho_0 g \int_0^{2\pi} \int_0^R (R^2 r - 2R^2 + r^3) \, dr \, d\theta
\]  
\[
= \frac{1}{2} \rho_0 g \int_0^{2\pi} \frac{1}{2} R^2 r^2 - \frac{2}{3} R^3 + \frac{1}{4} r^4 \bigg|_0^R d\theta = \frac{1}{24} \rho_0 g R^4 \int_0^{2\pi} d\theta = \frac{\pi \rho_0 g R^4}{12}
\]  

and from this we get \( \bar{z} \):

\[
\bar{z} = \frac{\pi \rho_0 g R^4}{12} \times \frac{3 \rho_0 g R^3 \pi}{12} = \frac{R}{4}
\]  

9.4 Center of gravity of an area

We have already done all of the heavy lifting necessary to compute the center of gravity for any type of object. Let us specialize this now for an object that is essentially two-dimensional:

where the object has a density \( \rho(x, y) \) and a thickness \( t(x, y) \). Then the total weight of the volume is

\[
w = \int_{\Omega} \rho(x, y) \, t(x, y) \, g \, dA
\]  

integrated over the area. The coordinates of the centroid are given by

\[
\bar{x} = \frac{1}{w} \int_{\Omega} x \rho(x, y) \, t(x, y) \, g \, dA
\]  
\[
\bar{y} = \frac{1}{w} \int_{\Omega} y \rho(x, y) \, t(x, y) \, g \, dA
\]  

Example 9.7

Compute the centroid of the following plate with constant thickness and density:
The weight of the volume is computed via the above formula:

\[
\begin{align*}
    w &= \int_\Omega \rho_0 T g \, dA = \int_0^L \int_0^{H \sqrt{x/L}} \rho_0 T g \, dy \, dx = \rho_0 T g \int_0^L \left. \frac{H \sqrt{x/L}}{L^{1/2}} \rho_0 T g \, dx \right|_0^L \\
    &= \frac{\rho_0 T g H}{L^{1/2}} \int_0^L x^{1/2} \, dx = \frac{\rho_0 T g H}{L^{1/2}} \frac{2x^{3/2}}{3} \bigg|_0^L = \frac{2}{3} \rho_0 T g H L
\end{align*}
\]  

(9.60)

To get \( \bar{x} \) we compute the moment integral first:

\[
\int_{\Omega} x \rho_0 T g \, dA = \rho_0 T g \int_0^L \int_0^{H \sqrt{x/L}} x \, dy \, dx = \frac{\rho_0 T g H}{L^{1/2}} \int_0^L x^{3/2} \, dx = \frac{\rho_0 T g H}{L^{1/2}} \frac{2L^{5/2}}{5} = \frac{2}{5} \rho_0 T g H L^2
\]

(9.62)

now we simply divide:

\[
\bar{x} = \frac{2}{5} \rho_0 T g H L^2 \times \frac{3}{2\rho_0 T g H L} = \frac{3L}{5}
\]

(9.63)

Similarly, to get \( \bar{y} \) we compute the moment integral:

\[
\int_{\Omega} y \rho_0 T g \, dA = \rho_0 T g \int_0^L \int_0^{H \sqrt{x/L}} y \, dy \, dx = \frac{\rho_0 T g H}{2} \int_0^L y^2 \bigg|_0^L \frac{H \sqrt{x/L}}{L^{1/2}} \, dx = \frac{\rho_0 T g H^2}{2L} \int_0^L x \, dx
\]

(9.64)

\[
= \frac{\rho_0 T g H^2 L^2}{4L} = \frac{\rho_0 T g H^2 L}{4}
\]

(9.65)

and again, we simply divide:

\[
\bar{y} = \frac{\rho_0 T g H^2 L}{4} \times \frac{3L}{2\rho_0 T g H L} = \frac{3H}{8}
\]

(9.66)

### 9.5 Center of gravity of a curve

Finally, let us consider the center of gravity of a long, slender, curved element such as a wire or a pipe. Here we will consider 2D curves only, although the 3D case follows easily by analogy.

As previously, we parameterize our curve by a distance vector:

\[
r(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \quad a \leq t \leq b
\]

(9.67)
Suppose the curve has a density (in mass per unit length) of \( \rho(t) \). Then the total weight of the curve is

\[
\mathbf{w} = \int_{\alpha}^{\beta} \rho(t) g \sqrt{x'^2(t) + y'^2(t)} \, dt
\]  

(9.68)

We can then compute the location of the centroid as

\[
\bar{x} = \frac{1}{\mathbf{w}} \int_{\alpha}^{\beta} x(t) \rho(t) g \sqrt{x'^2(t) + y'^2(t)} \, dt, \quad \bar{y} = \frac{1}{\mathbf{w}} \int_{\alpha}^{\beta} y(t) \rho(t) g \sqrt{x'^2(t) + y'^2(t)} \, dt
\]  

(9.69)

Note that we use \( x(t) \) and \( y(t) \) – we can't just substitute \( x \) and \( y \) because our integral is in terms of \( t \). Also note that \( \bar{x} \) and \( \bar{y} \) may not be located on or in the actual curve.

### 9.6 Composite bodies

Sometimes we need to find the centroid of an object that is composed of multiple subregions for which it is quite easy to find the centroid. For instance, what is the center of gravity for this object composed of several rectangular regions?

If we know the total weights \( w_1, w_2, \ldots \) and centroids \( (\bar{x}_1, \bar{y}_1), (\bar{x}_2, \bar{y}_2), \ldots \) of each element, then the total weight is simply

\[
\mathbf{w}_{\text{tot}} = \sum_{i} w_i
\]  

(9.70)

and the centroid coordinates are computed by an easy discrete sum:

\[
\bar{x}_{\text{tot}} = \frac{1}{\mathbf{w}_{\text{tot}}} \sum_{i} \bar{x}_i w_i, \quad \bar{y}_{\text{tot}} = \frac{1}{\mathbf{w}_{\text{tot}}} \sum_{i} \bar{y}_i w_i
\]  

(9.71)

#### Example 9.8

Consider the following body that is composed of a region with constant density \( \rho_1 \) and another region with density \( \rho_2 \).
From the previous example we know that if the entire region were the same density we would have

\[
\begin{align*}
\mathbf{w}_1 &= \frac{2\rho_1 T g H L}{3} \\
\bar{x}_1 &= \frac{3}{5}L \\
\bar{y}_1 &= \frac{3}{8}H
\end{align*}
\]

We can easily compute the weight and centroid of the second region,

\[
\begin{align*}
\mathbf{w}_2 &= \frac{L}{2} \times \frac{H}{2} \times \rho_2 T g = \frac{\rho_2 T g L H}{4} \\
\bar{x}_2 &= \frac{3L}{4} \\
\bar{y}_2 &= \frac{W}{4}
\end{align*}
\]

With this information, how can we compute the combined centroid?

Note that we need to “remove” a box-shaped section of the curved region before “replacing” it with a section of a different density. The total weight then is

\[
\mathbf{w}_{\text{tot}} = \frac{2\rho_1 T g H L}{3} - \frac{\rho_1 T g L H}{4} + \frac{\rho_2 T g L H}{4} = \left(\frac{5}{12}\rho_1 + \frac{1}{4}\rho_2\right) T g H L
\]

The x component of the centroid is simply

\[
\bar{x} = \frac{1}{\left(\frac{5\rho_1}{12} + \frac{\rho_2}{4}\right) T g H L} \left(\frac{3}{5}L \times \frac{4\rho_1 T g H L}{3} + \frac{3L}{4} \times \frac{(\rho_2 - \rho_1) T g L H}{4}\right)
\]

\[
\bar{x} = \frac{L}{\left(\frac{5\rho_1}{12} + \frac{\rho_2}{4}\right)} \left(\frac{2\rho_1}{5} + \frac{3(\rho_2 - \rho_1)}{16}\right)
\]

\[
\bar{x} = \frac{L(17\rho_1/80 + 3\rho_2/16)}{(5\rho_1/12 + \rho_2/4)}
\]

and a similar procedure can be followed for the y component.
Lecture 22  Center of Mass, Moment of Inertia

As a brief review: in the previous section we have introduced the formulae for computing the centroid of an object:

\[ w = \int_{\Omega} \rho g \, dv \quad \bar{x} = \frac{1}{w} \int_{\Omega} \rho g x \, dv \quad \bar{y} = \frac{1}{w} \int_{\Omega} \rho g y \, dv \quad \bar{z} = \frac{1}{w} \int_{\Omega} \rho g z \, dv \]  \quad (9.78)

where \( w \) is the weight, \( \bar{x}, \bar{y}, \bar{z} \) are the centroidal coordinates, and \( \int_{\Omega} \ldots \, dv \) is an integral over the entire object, which may be a volume, plate, line, etc.

### 9.7 Surface of Revolution

Let us consider objects that are axisymmetric and can be parameterized in the following way:

![Diagram](image)

How can we compute integrals over this surface? A convenient way of doing this integral is to compute \( da \) for a differential ring element. Then we can represent our integral as

\[ \int_{\Omega} da = \int_{a}^{b} da(z) \]  \quad (9.79)

We want to integrate over \( z \) only, so how can we represent \( da \) in terms of \( dz \)?

\[ da = [\text{length of ring}] \times [\text{height of ring}] = [2\pi r] \times [\sqrt{dr^2 + dz^2}] = [2\pi r] \times [\sqrt{1 + \left(\frac{dr}{dz}\right)^2} \, dz] \]

\[ = 2\pi r(z) \sqrt{1 + r'^2(z)} \, dz \]  \quad (9.80)

Consequently, we have that

\[ w = 2\pi \int_{a}^{b} \rho(z) g(z) r(z) \sqrt{1 + r'^2(z)} \, dz \quad \bar{z} = \frac{2\pi}{w} \int_{a}^{b} z \rho(z) g(z) r(z) \sqrt{1 + r'^2(z)} \, dz \]  \quad (9.81)

Because the problem is axisymmetric, we see immediately that \( \bar{x} = \bar{y} = 0 \)

**Example 9.9**

Compute the centroid of the following cone with constant mass per unit area \( \rho_0 \) in a gravitational field with acceleration \( g \).
First, we compute the total weight of the cone:

\[ w = \int_{\Omega} \rho_0 g da = \rho_0 g \int_{0}^{H} 2\pi r(z) \sqrt{1 + r'^2(z)} dz = 2\pi \rho_0 g \int_{0}^{H} \frac{Rz}{H} \sqrt{1 + (R/H)^2} dz \]

\[ = 2\pi \rho_0 g R \frac{\sqrt{1 + (R/H)^2}}{H} \int_{0}^{H} z dz = \frac{2\pi \rho_0 g R \sqrt{1 + (R/H)^2}}{H} \times \frac{H^2}{2} = \pi \rho_0 g R \sqrt{1 + (R/H)^2} \]  \hspace{1cm} (9.82)

Now, compute \( \bar{z} \):

\[ \bar{z} = \frac{1}{w} \int_{\Omega} \rho_0 g z da = \frac{\rho_0 g}{w} \int_{0}^{H} 2\pi r(z) \sqrt{1 + r'^2(z)} dz = 2\pi \rho_0 g \frac{\sqrt{1 + (R/H)^2}}{wH} \int_{0}^{H} z^2 dz \]

\[ = 2\pi \rho_0 g R \frac{\sqrt{1 + (R/H)^2}}{wH} \int_{0}^{H} \frac{Rz^2}{H} \sqrt{1 + (R/H)^2} dz \]

\[ = 2\pi \rho_0 g R \frac{\sqrt{1 + (R/H)^2}}{wH} \times \frac{H^3}{3} = \frac{2\pi \rho_0 g R H \sqrt{1 + (R/H)^2}}{3w} \]  \hspace{1cm} (9.83)

\[ = \frac{2\pi \rho_0 g R H \sqrt{1 + (R/H)^2}}{3} \]

\[ = \frac{2}{3} H \]  \hspace{1cm} (9.84)

9.8 Centroid Frame Invariance

In our derivation of the equations for \( \bar{x}, \bar{y}, \bar{z} \), we were always using the origin as our reference point. What if we decided to use a different coordinate system instead? For instance, let \( x' = x + x_0, y' = y + y_0 \):

Then we have

\[ dx' = dx(x + x_0) = dx \]
\[ dy' = dy(y + y_0) = dy \]
\[ \implies da' = dx'dy' = dx dy = da \]  \hspace{1cm} (9.85)

so our weight integral is

\[ w' = \int_{\Omega} \rho(x' - x_0, y' - y_0) g(x' - x_0, y' - y_0) t(x' - x_0, y' - y_0) da' = \int_{\Omega} \rho(x, y) g(x, y) t(x, y) da = w \]  \hspace{1cm} (9.86)
in other words, the value of the weight is the same regardless of what coordinates we use. What happens when we compute the centroid?

\[
\bar{x}' = \frac{1}{w} \int_{\Omega} x' \rho g \, da' = \frac{1}{w} \int_{\Omega} (x + x_0) \rho g \, da = \frac{1}{w} \int_{\Omega} x \rho g \, da + \frac{x_0}{w} \int_{\Omega} \rho g \, da = \bar{x} + x_0
\]  

(9.90)

which means that the location of the centroid in the object is independent of the coordinate system we use.

### 10 Moment of Inertia

Here we take a brief segue to introduce a topic that is related to (but different from) the computation of centroids in a 2D or 3D body. Moments of inertia of a body frequently pop up in:

- Rotational motion laws (you'll see these in Dynamics)
- Stresses in a beam (you'll see these in Mechanics of Materials)

Unfortunately, we don't have enough machinery at this point to truly motivate the need for moments of inertia. But, for some reason, computing moments of inertia is a standard part of Statics, so we will introduce them and learn how to compute them, and you'll have to take my word for it that it will be useful later.

#### 10.1 2D moment of inertia

Before beginning, I would like to clear up some notation. As frequently happens in old fields, a lot of the notation has become dated. The following expressions are, generally, equivalent:

- Area moment of inertia
- Second moment of area

Consider some 2D region.

Let us define the following quantities:

\[
l_{xx} = \int_{\Omega} y^2 \, da \quad l_{yy} = \int_{\Omega} x^2 \, da \quad l_{xy} = \int_{\Omega} x y \, da
\]  

(10.1)

where

- \( l_{xx} \) is the 2nd moment of area about the x axis
- \( l_{yy} \) is the 2nd moment of area about the y axis
- \( l_{xy} \) is the product moment of area

(Note: these quantities are purely geometrical. Therefore it is a bit misleading to refer to them as moments of inertia.)
Example 10.1

Compute the moments of inertia for the following with the axes in two different locations:

For the axis at the centroid:

\[
I_{xx} = \int_{-W/2}^{W/2} \int_{-H/2}^{H/2} y^2 \, dy \, dx = \int_{-W/2}^{W/2} \frac{y^3}{3} \bigg|_{-H/2}^{H/2} \, dx = \frac{H^3}{12} \int_{-W/2}^{W/2} \, dx = \frac{H^3 W}{12} \tag{10.2}
\]

\[
I_{yy} = \int_{-H/2}^{H/2} \int_{-W/2}^{W/2} x^2 \, dx \, dy = \int_{-H/2}^{H/2} \frac{x^3}{3} \bigg|_{-W/2}^{W/2} \, dy = \frac{W^3}{12} \int_{-H/2}^{H/2} \, dy = \frac{W^3 H}{12} \tag{10.3}
\]

\[
I_{xy} = \int_{-W/2}^{W/2} \int_{-H/2}^{H/2} y \, dx \, dy = \int_{-W/2}^{W/2} \frac{y^2}{2} \bigg|_{-H/2}^{H/2} \, dx = 0 \tag{10.4}
\]

For the axis at the lower left corner:

\[
I_{xx} = \int_{0}^{W} \int_{0}^{H} y^2 \, dy \, dx = \int_{0}^{W} \frac{y^3}{3} \bigg|_{0}^{H} \, dx = \frac{H^3}{3} \int_{0}^{W} \, dx = \frac{H^3 W}{3} \tag{10.5}
\]

\[
I_{xx} = \int_{0}^{H} \int_{0}^{W} x^2 \, dx \, dy = \int_{0}^{H} \frac{x^3}{3} \bigg|_{0}^{W} \, dy = \frac{W^3 H}{3} \tag{10.6}
\]

\[
I_{xy} = \int_{0}^{W} \int_{0}^{H} x \, dy \, dx = \int_{0}^{W} \frac{x^2}{2} \bigg|_{0}^{H} \, dx = \frac{H^2}{2} \int_{0}^{W} \, dx = \frac{H^2 W^2}{4} \tag{10.7}
\]

10.2 Physical interpretation

Now that we've learned about the moment of inertia and performed a calculation, let's take a quick look at the role of the moment of inertia in dynamics and mechanics of materials:

10.2.1 Conservation of angular momentum

In dynamics, conservation of momentum acts analogously to mass in the equation for conservation of angular momentum:

\[
M = \frac{d}{dt}(\rho I \omega) \tag{10.8}
\]

where \(M\) is the applied moment and \(\omega\) is the angular velocity. (Note that this is the scalar version of the conservation law; \(M\) and \(\omega\) are usually vectors.) In the above case, the two moments of inertia corresponded to rotation about the two different axes:
Note that it is much "harder" to rotate about the second axis—this corresponds with the result that the moment of inertia in the second case is much greater.

### 10.2.2 Stress in a beam

Consider a beam subjected to a load as shown:

Then the maximum stress in the beam is given by

$$\sigma_{\text{max}} = \frac{M c}{I}$$

(10.9)

where $I$ is the moment of inertia in the $x$ direction about the centroid. Notice that a large moment of inertia corresponds to a low stress, so beams are generally designed to have a high moment of inertia.

### 10.3 Parallel Axis Theorem

Recall how we showed that the centroid location is irrelevant of location. We'll see that the same is not true for moment of inertia.

Consider the following body

Let us set our coordinate system in such a way that the origin is located at the centroid of the object, in other words, $\bar{x} = \bar{y} = 0$.

Now, as before, let us define another set of coordinates $x' = x + x_0, y' = y + y_0$. Let us compute the moments of area about the $x'$ and $y'$ axes:

$$I'_{xx} = \int_{\Omega} (y')^2 \, da' = \int_{\Omega} (y + y_0)^2 \, da = \int_{\Omega} y^2 \, da + 2y_0 \int_{\Omega} y \, da + y_0^2 \int_{\Omega} \, da$$

$$= I_{xx} + y_0^2 A = I'_{xx}$$

(10.10)
where \( I_{xx} \) is the 2nd moment of area about the centroidal x axis, \( y_0 \) is the distance to the centroidal x axis, and \( A \) is the area of the region. Similarly we get that

\[
I'_{yy} = I_{yy} + x_0^2 A
\]  

(10.11)

Finally, we have

\[
l'_{xy} = \int_{\Omega} (x') (y') \, da' = \int_{\Omega} (x + x_0) (y + y_0) \, da = \int_{\Omega} xy \, da + x_0 \int_{\Omega} y \, da + y_0 \int_{\Omega} x \, da + x_0 y_0 \int_{\Omega} da
\]  

(10.12)

\[
= I_{xy} + x_0 y_0 A = l'_{xy}
\]  

(10.13)

**Example 10.2**

Compute the moment of inertia about the centroidal x axis:

There are two ways to do this. Let us start with the parallel axis theorem. From the previous example we know that

\[
l^1_{xx} = \frac{h^3 L}{12} \quad l^2_{xx} = \frac{L^3 h}{12} \quad l^3_{xx} = \frac{h^3 L}{12}
\]  

(10.14)

where \( l^{1,2,3}_{xx} \) are the moments of inertia of each of the three sections about their centroidal x axes. The areas and distances are

\[
A^1 = A^2 = A^3 = L h \quad y_1^0 = \frac{1}{2} (L + h) \quad y_0^2 = 0 \quad y_0^3 = -\frac{1}{2} (L + h)
\]  

(10.15)

The moment of inertia, computed using the parallel axis theorem, is:

\[
l_{xx} = l^1_{xx} + (y_1^0)^2 A^1 + l^2_{xx} + (y_0^2)^2 A^2 + l^3_{xx} + (y_0^3)^2 A^3
\]  

(10.16)

\[
= \frac{L h^3}{12} + \left( \frac{L + h}{2} \right)^2 (L h) + \frac{L^3 h}{12} + \left( - \frac{L + h}{2} \right)^2 (L h)
\]  

(10.17)

\[
= 2 \left( \frac{L h^3}{12} + \frac{L^3 h + 2L^2 h^2 + Lh^3}{4} \right) + \frac{L^3 h}{12}
\]  

(10.18)

\[
= \frac{2L h^3}{3} + L^2 h^2 + \frac{7L^3 h}{12}
\]  

(10.19)
Alternatively, we could take the moment of inertia for the large block and then subtract two smaller regions:

\[I_{xx} = \left( \frac{(L + 2h)^3 L}{12} \right)_{\text{total}} - 2 \times \left( \frac{L^3((L - h)/2)}{12} \right)_{\text{holes}}\]

\[= \frac{L^4 + 6L^3h + 12L^2h^2 + 8Lh^3}{12} - \frac{L^4 - L^3h}{12}\]  

\[= \frac{2Lh^3}{3} + L^2h^2 + \frac{7L^3h}{12}\]  

(10.20)  

(10.21)  

(10.22)

giving us the same result.
10.4 Polar moment of inertia

So far we have defined the moments of inertia about the x axis and the y axis, and we have also defined the “product of inertia.” These are useful for defining laws of motion (assuming constant density)

\[
M_x = \frac{d}{dt}(\rho I_{xx}\omega_x) \quad M_y = \frac{d}{dt}(\rho I_{yy}\omega_y)
\]

(10.23)

where \(M_x, M_y\) are moments about the x and y axes, and \(\omega_x, \omega_y\) are the angular velocities about the x and y axes.

(Note: the above are only true when \(I_{xy} = 0\) ...we’ll see why in a minute)

What if I want to find a relationship between \(M_z\) and \(\omega_z\)? That is, what if I am looking at this thing from the top, and I want to spin it in-plane?

\[
\int_{\Omega} r^2 \, da
\]

(10.24)

where \(r = \sqrt{x^2 + y^2}\). We can simplify the above:

\[
\int_{\Omega} r^2 \, da = \int_{\Omega} (x^2 + y^2) \, da = \int_{\Omega} x^2 \, da + \int_{\Omega} y^2 \, da = I_{yy} + I_{xx} = l_{zz}
\]

(10.25)

so we have

\[
M_z = \frac{d}{dt}(\rho l_{zz}\omega_z)
\]

(10.26)

Important note: the relationship \(l_{zz} = l_{xx} + l_{yy}\) is only true in 2D, as we will see in the following section.

10.5 Moment of inertia in 3D

Previously we have considered the moment of inertia for an area, that is, we were looking at the moment for a 2D region. Now, let us consider the moment of inertia for a 3D volume:
Previously we had three versions of our moment of inertia: \( I_{xx}, I_{yy}, I_{xy} \) (not counting the polar moment). Now, in three dimensions, we have six:

\[
\begin{align*}
I_{xx} &= \int_{\Omega} (y^2 + z^2) \, dv \\
I_{yy} &= \int_{\Omega} (z^2 + y^2) \, dv \\
I_{zz} &= \int_{\Omega} (x^2 + y^2) \, dv \\
I_{xy} &= \int_{\Omega} xy \, dv \\
I_{yx} &= \int_{\Omega} yx \, dv \\
I_{zx} &= \int_{\Omega} zx \, dv
\end{align*}
\] (10.27)

Note that I recover the same results for 2D: that is, if \( \Omega \) is 2D and \( z = 0 \) then I have

\[
\begin{align*}
I_{xx} &= \int_{\Omega} y^2 \, dv \\
I_{yy} &= \int_{\Omega} y^2 \, dv \\
I_{zz} &= \int_{\Omega} (x^2 + y^2) \, dv \\
I_{xy} &= 0 \\
I_{yx} &= 0
\end{align*}
\] (10.29)

What is the relationship between all of these quantities? To understand this relationship, we need to introduce tensors.

### 10.6 Introduction to tensors

In introductory courses, we are introduced to scalar quantities, which allow us to solve simplified equations generally in one dimension. We learn that a scalar is insufficient to represent things like force, acceleration, etc, so we represent those quantities in terms of vectors. In mechanics (and other fields as well) we frequently encounter quantities that cannot be represented as a vector only. For instance, let us consider the forces that we encountered in beam theory, \( N(x), V(x) \).

Let us generalize this to an arbitrary square-shaped region. How do we represent all of the forces (or stresses) acting on it? We use the stress tensor:

\[
\sigma = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix}
\] (10.31)

(we note that \( \sigma_{xy} = \sigma_{yx} \) for conservation of angular momentum, making the stress tensor symmetric). Note that there are now two notions of directionality: there is the direction of the force and the direction of the face on which it is acting.

#### 10.6.1 Tensors as operators on vectors

One useful way to think about a tensor is as a machine that turns one vector into another. We can think of the stress tensor as a machine that turns the normal vector of a surface into the force vector (per unit area) acting on the surface. For instance, if we multiply the stress tensor above by the normal vector corresponding to the right and left faces:

\[
\begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \sigma_{xx} \\ \sigma_{yx} \end{bmatrix}
\]

\[
\begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\sigma_{xx} \\ -\sigma_{yx} \end{bmatrix}
\] (10.32)

#### 10.6.2 Invariance of tensors

Consider a vector.
What are the coordinates of this vector? *It depends on our choice of coordinate system.* Our vector will be the same regardless of our choice of coordinates, we just change the components accordingly. Similarly, a tensor is the same regardless of our choice of coordinates. Therefore, we say that a tensor *transforms* when we change coordinates.

### 10.7 Moment of Inertia Tensor

With our new tensor technology, we have the tools we need to define the *moment of inertia tensor*. It is defined as:

\[
\mathbf{I} = \begin{bmatrix}
I_{xx} & -I_{xy} & I_{xz} \\
-I_{xy} & I_{yy} & -I_{yz} \\
-I_{xz} & -I_{yz} & I_{zz}
\end{bmatrix}
\]  

(10.33)

where the components are the 3D moments of inertia as described above. In 2D we have

\[
\mathbf{I} = \begin{bmatrix}
I_{xx} & -I_{xy} \\
-I_{xy} & I_{yy}
\end{bmatrix}
\]  

(10.34)

Unfortunately the derivation (including why we have those mysterious negative signs) is beyond the scope of this class. However, we will still be able to do some interesting things with it.

#### 10.7.1 Applications to Dynamics

Recall the equations of motion that we described earlier, using scalar moments and angular velocities. But we know that moments are really vectors, which means we should be able to write a vector equation for angular momentum: To do this, we need to define the vector angular velocity: \( \mathbf{\omega} \).

What is the relationship between \( \mathbf{M} \) and \( \mathbf{\omega} \)?

\[
\mathbf{M} = \frac{d}{dt}(\rho \mathbf{I} \mathbf{\omega}) = \begin{bmatrix}
\rho I_{xx} \dot{\omega}_x & -\rho I_{xy} \dot{\omega}_y & \rho I_{xz} \dot{\omega}_z \\
-\rho I_{xy} \dot{\omega}_x & \rho I_{yy} \dot{\omega}_y & -\rho I_{yz} \dot{\omega}_z \\
-\rho I_{xz} \dot{\omega}_x & -\rho I_{yz} \dot{\omega}_y & \rho I_{zz} \dot{\omega}_z
\end{bmatrix}
\]  

(10.35)

where \( \dot{\omega}_x, \dot{\omega}_y, \dot{\omega}_z \) are the derivatives of the components of the angular velocity vector.

Things to note:

- The direction of the moment and the direction of the angular acceleration are not necessarily parallel.
- The tensor is symmetric.
- When do we know that they are parallel? Stay tuned...
10.8 Moment of inertia about a specific axis

Let us suppose that we select an arbitrary unit vector \( \mathbf{n} \). Then let us constrain our moments and angular acceleration such that \( \mathbf{M} = M \mathbf{n} \), \( \omega = \omega \mathbf{n} \). Let us substitute this into the above equation:

\[
M \mathbf{n} = \frac{d}{dt}(\rho \mathbf{I} \omega \mathbf{n})
\]

(10.36)

Dot both sides with \( \mathbf{n} \) to obtain

\[
M (\mathbf{n} \cdot \mathbf{n}) = \mathbf{n} \cdot \frac{d}{dt}(\rho \mathbf{I} \omega \mathbf{n}) \implies M = \frac{d}{dt}(\rho (\mathbf{n} \cdot \mathbf{I} \mathbf{n}) \omega)
\]

(10.37)

where we define the scalar quantity

\[
\mathbf{I}_n = \mathbf{n} \cdot \mathbf{I} \mathbf{n}
\]

(10.38)

as the moment of inertia about \( \mathbf{n} \).

**Example 10.3**

Consider the beam that we computed in the previous lecture. Find the moment of inertia tensor for the given axes, and the moment of inertia about an axis inclined at an angle \( \theta \) as shown:

![Beam Diagram]

To compute our moment of inertia tensor, perform the integrals as before:

\[
l_{xx} = \int_{-W/2}^{W/2} \int_{-H/2}^{H/2} y^2 dy \, dx = \frac{WH^3}{12}
\]

(10.39)

\[
l_{yy} = \int_{-W/2}^{W/2} \int_{-H/2}^{H/2} x^2 dy \, dx = \frac{HW^3}{12}
\]

(10.40)

\[
l_{xy} = \int_{-W/2}^{W/2} \int_{-H/2}^{H/2} xy dy \, dx = 0
\]

(10.41)

\[
l_{zz} = l_{xx} + l_{yy} = \frac{H^3W + HW^3}{12}
\]

(10.42)

So our moment of inertia tensor is

\[
\mathbf{I} = \begin{bmatrix}
\frac{1}{12}H^3W & 0 & 0 \\
0 & \frac{1}{12}HW^3 & 0 \\
0 & 0 & \frac{1}{12}(H^3W + HW^3)
\end{bmatrix}
\]

(10.43)

Now, we need to find a vector \( \mathbf{n} \) that runs along our dotted line. It will be the familiar

\[
\mathbf{n} = \begin{bmatrix}
\cos \theta \\
\sin \theta \\
0
\end{bmatrix}
\]

(10.44)
Substituting, we get
\[ I_n = \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ \cos \theta \sin \theta & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{12} H^3 W & 0 & 0 \\ 0 & \frac{1}{12} H W^3 & 0 \\ 0 & 0 & \frac{1}{12} (H^3 W + HW^3) \end{bmatrix} \begin{bmatrix} \cos \theta \\ \sin \theta \\ 0 \end{bmatrix} \] (10.45)

\[ = \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ \cos \theta \sin \theta & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{12} H^3 W \cos \theta & 0 & 0 \\ \frac{1}{12} H W^3 \sin \theta & 0 \end{bmatrix} \] (10.46)

\[ = \frac{H^3 W \cos^2 \theta + HW^3 \sin^2 \theta}{12} \] (10.47)

Note that we recover \( I_{xx} \) for \( \theta = 0 \) and \( I_{yy} \) for \( \theta = (\pi/2) \).

### 10.9 Eigenvalues and eigenvectors

Earlier, we asked the question: under what conditions is the angular acceleration the same direction as the applied moment? In other words, for what directions \( n \) do we have (for constant density)

\[ M n = \rho I \omega n \] (10.48)

To answer this question, we need to introduce eigenvalues and eigenvectors.

Suppose we have a matrix \( A \):

\[ A = \begin{bmatrix} A_{xx} & A_{xy} & A_{xz} \\ A_{yx} & A_{yy} & A_{yz} \\ A_{zx} & A_{zy} & A_{zz} \end{bmatrix} \] (10.49)

For what values of \( n \) do we get

\[ An = \lambda n \] (10.50)

where \( \lambda \) is a scalar. In other words, given a matrix, is there a vector that does not change direction when multiplied by it? To find out, we will simply solve the above equation:

\[ An = \lambda I = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} n \Rightarrow \begin{bmatrix} A_{xx} - \lambda & A_{xy} & A_{xz} \\ A_{yx} & A_{yy} - \lambda & A_{yz} \\ A_{zx} & A_{zy} & A_{zz} - \lambda \end{bmatrix} \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \] (10.51)

where \( I \) is the identity matrix. Now what? It turns out that in order for the above equation to be true (without just making \( n = 0 \)) we must have \( \det(A - \lambda I) \), which gives us

\[ (A_{xx} - \lambda)(A_{yy} - \lambda)(A_{zz} - \lambda) + A_{xy}A_{yz}A_{xz} + A_{xz}A_{yz}A_{xy} - (A_{xx} - \lambda)A_{yx}A_{zy} - A_{xz}A_{yz}A_{xy} - A_{xy}A_{yz}A_{xz} - \lambda)A_{xx} - \lambda = 0 \] (10.52)

This is called the characteristic equation and is a third order cubic equation that we can use to solve for \( \lambda \). Because it is third order, we know that we will have as many as three possible solutions. This means we will get as many as three possible vectors satisfying \( An = \lambda n \). The \( \lambda \) are called eigenvalues and the \( n \) are called eigenvectors.

**Example 10.4**

Compute the eigenvalues and eigenvectors of the following tensor:

\[ A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \] (10.53)

First step: compute the determinant of the matrix with \( \lambda \) subtracted from all of the diagonal terms, as done in...
Equation 10.52:
\[
\det \begin{bmatrix}
1 - \lambda & 2 & 0 \\
2 & 1 - \lambda & 0 \\
0 & 0 & 1 - \lambda
\end{bmatrix} = (1 - \lambda)^3 - 4(1 - \lambda) = 0
\] (10.54)

Next step: solve for lambda. Remember, we will probably have multiple values, so we need to find all of them.

\[
(1 - \lambda)[(1 - \lambda)^2 - 4] = 0 \implies (1 - \lambda) = 0 \implies \lambda_1 = 1
\] (10.55)
\[
(1 - \lambda)^2 - 4 = 0
\] (10.56)
\[
1 - \lambda = \pm 2
\] (10.57)
\[
\implies \lambda_2 = -1, \lambda_3 = 3
\] (10.58)

So we have three possible eigenvalues, 1, -1, 3. This means we have three eigenvectors. We can solve for these using Equation 10.51:

\[
\lambda_1 = 1 : \begin{bmatrix}
1 - (1) & 2 & 0 \\
2 & 1 - (1) & 0 \\
0 & 0 & 1 - (1)
\end{bmatrix} \begin{bmatrix}
x \\
y \\
z
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\] (10.59)
\[
\implies n_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\] (10.60)

Notice how the solution for \( n_1 \) is nonunique – that is, it could be any magnitude we want. Because we like unit vectors, we choose one so that the magnitude is unity. We follow a similar procedure to find the remaining eigenvectors.

\[
\lambda_1 = -1 : \begin{bmatrix}
1 - (-1) & 2 & 0 \\
2 & 1 - (-1) & 0 \\
0 & 0 & 1 - (-1)
\end{bmatrix} \begin{bmatrix}
x \\
y \\
z
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\] (10.61)
\[
\implies n_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}
\] (10.62)

\[
\lambda_1 = -1 : \begin{bmatrix}
1 - (3) & 2 & 0 \\
2 & 1 - (3) & 0 \\
0 & 0 & 1 - (3)
\end{bmatrix} \begin{bmatrix}
x \\
y \\
z
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\] (10.63)
\[
\implies n_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\] (10.64)

Recipe for computing eigenvalues and eigenvectors:

1. Find the characteristic polynomial \( \det(A - \lambda I) = 0 \) where \( I \) is the identity matrix.
2. Solve the characteristic polynomial for \( \lambda \) – for an \( n \times n \) matrix. (You may have up to \( n \) values.)
3. For each \( \lambda_i \), substitute into \( (A - \lambda I)n = 0 \) to solve for \( n \). Remember that the solution will be nonunique.
4. Normalize the eigenvectors \( n \).

Notes:

- If a matrix is symmetric, its eigenvectors are all orthogonal to each other.
- The eigenvectors of the moment of inertia tensor are called principal axes.
11 Kinematics of a Particle

We are officially finished with the statics part of the course. Up until now we have assumed that there is no time dependence and that everything is in static equilibrium. In the remaining three lectures, we will begin to introduce the framework for non-equilibrium systems.

Definition 11.1. Kinematics is the construction of geometrically necessary quantities and relationships arising in the study of motion.

For example, velocity and acceleration are simply definitions; we do not need physical laws to study them. In kinematics, we study only quantities that can be measured directly.

Definition 11.2. Dynamics is the application of physical laws to the study of motion.

For example, Newton's second law relates a kinematic quantity (acceleration) to a dynamic quantity (force). In dynamics, in addition to kinematic quantities, we consider quantities that cannot be measured directly.

(Note: what does it mean to measure a quantity directly? Can we measure a force directly? What would such an experiment look like? In general, any measurement of force is done by measuring a displacement and then computing the total force.)

In the remainder of this course, we will study the kinematics of a particle. That is, we will define notions of position, velocity and acceleration for a particle in various situations. This is the dynamics analog of studying particle equilibrium and, just as with statics, we are able to simplify things considerably when dealing with a single point.

11.1 Change of basis

In statics, we can generally get by pretty easily using just Cartesian coordinates. When using Cartesian coordinates, we generally write vectors in the following way

\[ \mathbf{r} = \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} \]  

where \( r_x, r_y, r_z \) are components in the \( x, y, z \) directions. Another way to write this is

\[ \mathbf{r} = r_x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + r_y \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + r_z \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = r_x \mathbf{i} + r_y \mathbf{j} + r_z \mathbf{k} = r_x \mathbf{g}_x + r_y \mathbf{g}_y + r_z \mathbf{g}_z \]  

(11.2)

where \( \mathbf{i}, \mathbf{g}_x, \mathbf{g}_y, \mathbf{g}_z \) are unit vectors. From now on, I will drop the \( \mathbf{i}, \mathbf{j}, \mathbf{k} \) notation and stick with \( \mathbf{g}_x, \mathbf{g}_y, \mathbf{g}_z \).

The set of vectors \( \mathbf{g}_x, \mathbf{g}_y, \mathbf{g}_z \) have some special properties:

1. They are all unit (or normalized) vectors (magnitude 1), or \( \mathbf{g}_x \cdot \mathbf{g}_x = \mathbf{g}_y \cdot \mathbf{g}_y = \mathbf{g}_z \cdot \mathbf{g}_z = 1 \)

2. They are mutually orthogonal, or \( \mathbf{g}_y \cdot \mathbf{g}_x = \mathbf{g}_z \cdot \mathbf{g}_x = \mathbf{g}_x \cdot \mathbf{g}_y = 0 \)

These two properties imply a third:

3. You can construct any other vector using a linear combination of these vectors.

We say that any set of three vectors that satisfy the above properties are called an orthonormal basis for \( \mathbb{R}^3 \) (three dimensional space).
Are \( \mathbf{g}_x, \mathbf{g}_y, \mathbf{g}_z \) the only vectors that form an orthonormal basis? No: for example, consider the following basis

\[
\mathbf{g}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \mathbf{g}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \end{bmatrix} \quad \mathbf{g}_3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\] (11.3)

These form a complete basis, which means that for any vector \( \mathbf{r} \), we can write

\[
\mathbf{r} = r_1 \mathbf{g}_1 + r_2 \mathbf{g}_2 + r_3 \mathbf{g}_3
\] (11.4)

How do we compute the coefficients \( r_1, r_2, r_3 \)? We can use the orthogonality property to extract them in the following way: dot both sides by \( \mathbf{g}_1 \)

\[
\mathbf{r} \cdot \mathbf{g}_1 = r_1 \mathbf{g}_1 \cdot \mathbf{g}_1 + r_2 \mathbf{g}_2 \cdot \mathbf{g}_1 + r_3 \mathbf{g}_3 \cdot \mathbf{g}_1 = r_1
\] (11.5)

Similarly, \( r_2 = \mathbf{r} \cdot \mathbf{g}_2, r_3 = \mathbf{r} \cdot \mathbf{g}_3 \). Let’s try this with an example:

**Example 11.1**

Find the coefficients \( r_1, r_2, r_3 \) of \( \mathbf{r} \) in the basis \( \mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3 \), where

\[
\mathbf{r} = \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} \quad \mathbf{g}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \mathbf{g}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \end{bmatrix} \quad \mathbf{g}_3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\] (11.6)

in the above basis, and show that the original vector is recovered.

Using orthogonality, we get

\[
r_1 = \mathbf{r} \cdot \mathbf{g}_1 = [r_x \quad r_y \quad r_z] \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix} = \frac{r_x + r_y}{\sqrt{2}}
\] (11.7)

\[
r_2 = \mathbf{r} \cdot \mathbf{g}_2 = [r_x \quad r_y \quad r_z] \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} = \frac{-r_x + r_y}{\sqrt{2}}
\] (11.8)

\[
r_3 = \mathbf{r} \cdot \mathbf{g}_3 = [r_x \quad r_y \quad r_z] \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = r_z
\] (11.9)

so we can write our vector as

\[
\mathbf{r} = \left( \frac{r_x + r_y}{\sqrt{2}} \right) \mathbf{g}_1 + \left( \frac{-r_x + r_y}{\sqrt{2}} \right) \mathbf{g}_2 + r_z \mathbf{g}_3
\] (11.10)

Notice that this is exactly the same vector! We have just written it in terms of different bases. Let’s confirm this by plugging in our expressions for the basis vectors:

\[
\mathbf{r} = \begin{bmatrix} r_x + r_y \\ -r_x + r_y \\ r_z \end{bmatrix}
\] (11.11)

\[
= \begin{bmatrix} (r_x + r_y)/2 \\ (r_x + r_y)/2 \\ 0 \end{bmatrix} + \begin{bmatrix} (r_x - r_y)/2 \\ (r_x - r_y)/2 \\ 0 \end{bmatrix} + r_z \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\] (11.12)

Exactly as we expected.

Let us represent this pictorially:
Notice how it is the same vector, just represented using different basis vectors.

### 11.2 Basis vectors in curvilinear coordinates

Let us define the position vector

\[ \mathbf{r}(x, y, z) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \]  

(11.13)

We can construct the three unit vectors using the following formula:

\[
\begin{align*}
\mathbf{g}_x &= \frac{1}{||\partial \mathbf{r} / \partial x||} \partial \mathbf{r} / \partial x \\
\mathbf{g}_y &= \frac{1}{||\partial \mathbf{r} / \partial y||} \partial \mathbf{r} / \partial y \\
\mathbf{g}_z &= \frac{1}{||\partial \mathbf{r} / \partial z||} \partial \mathbf{r} / \partial z
\end{align*}
\]  

(11.14)

that is, the unit vectors are the direction of change of the position with respect to the coordinates. One thing we note is that these vectors are all constant, that is,

\[
\frac{\partial}{\partial x} \mathbf{g}_{x,y,z} = \frac{\partial}{\partial y} \mathbf{g}_{x,y,z} = \frac{\partial}{\partial z} \mathbf{g}_{x,y,z} = 0
\]  

(11.15)

Now, let us apply this same procedure to other coordinate systems.

#### 11.2.1 Cylindrical coordinates

We can define the position vector as

\[ \mathbf{r}(\rho, \theta, z) = \begin{bmatrix} \rho \cos \theta \\ \rho \sin \theta \\ z \end{bmatrix} \]  

(11.16)

We can get our three basis vectors in exactly the same way:

\[
\begin{align*}
\mathbf{g}_\rho &= \frac{1}{\rho} \frac{\partial \mathbf{r}}{\partial \rho} = \begin{bmatrix} \cos \theta \\ \sin \theta \\ 0 \end{bmatrix} \\
\mathbf{g}_\theta &= \frac{\rho}{\rho} \frac{1}{\rho \cos \theta} \frac{\partial \mathbf{r}}{\partial \theta} = \begin{bmatrix} -\rho \sin \theta \\ \rho \cos \theta \\ 0 \end{bmatrix} \\
\mathbf{g}_z &= \frac{1}{\rho} \frac{\partial \mathbf{r}}{\partial z} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\end{align*}
\]  

(11.17)  

(11.18)  

(11.19)

Note that these bases are not constant. Let us compute the derivatives:

\[
\begin{align*}
\frac{\partial \mathbf{g}_\rho}{\partial \rho} &= 0 \\
\frac{\partial \mathbf{g}_\rho}{\partial \theta} &= 0 \\
\frac{\partial \mathbf{g}_\rho}{\partial z} &= 0 \\
\frac{\partial \mathbf{g}_\theta}{\partial \rho} &= 0 \\
\frac{\partial \mathbf{g}_\theta}{\partial \theta} &= -\mathbf{g}_\rho \\
\frac{\partial \mathbf{g}_\theta}{\partial z} &= 0 \\
\frac{\partial \mathbf{g}_z}{\partial \rho} &= 0 \\
\frac{\partial \mathbf{g}_z}{\partial \theta} &= 0 \\
\frac{\partial \mathbf{g}_z}{\partial z} &= 0
\end{align*}
\]  

(11.20)  

(11.21)  

(11.22)
11.2.2 Spherical Coordinates

Up until now we have, in general, avoided spherical coordinates as they are not usually needed for statics. However, they are quite relevant in dynamics, particularly for things like doing dynamics in a rotating spherical reference frame.

\[
\theta \quad \phi \\
ρ \\
x \quad y \quad z
\]

(Note: \( \theta \) and \( \phi \) are frequently reversed. I find this convention to be less confusing.)

To find our unit vectors, we construct our position vector:

\[
r(ρ, θ, φ) = \begin{bmatrix} ρ cos θ sin φ \\ ρ sin θ sin φ \\ ρ cos φ \end{bmatrix}
\]

Now we find the unit vectors by differentiation:

\[
\frac{∂r}{∂ρ} = \begin{bmatrix} cos θ sin φ \\ sin θ sin φ \\ cos φ \end{bmatrix} = \sqrt{sin^2 θ(cos^2 φ + sin^2 φ) + cos^2 θ} = 1
\]

\[
g_ρ = \begin{bmatrix} cos θ sin φ \\ sin θ sin φ \\ cos φ \end{bmatrix}
\]

\[
\frac{∂r}{∂θ} = \begin{bmatrix} -ρ sin θ sin φ \\ ρ cos θ sin φ \\ 0 \end{bmatrix} = \sqrt{ρ^2 sin^2 φ(sin^2 θ + cos^2 θ)} = ρ sin φ
\]

\[
g_θ = \begin{bmatrix} -sin θ \\ cos θ \\ 0 \end{bmatrix}
\]

\[
\frac{∂r}{∂φ} = \begin{bmatrix} ρ cos θ cos φ \\ ρ sin θ cos φ \\ -ρ sin φ \end{bmatrix} = \sqrt{ρ^2 cos^2 φ(sin^2 θ + cos^2 θ) + ρ^2 sin^2 φ} = ρ
\]

\[
g_φ = \begin{bmatrix} cos θ cos φ \\ sin θ cos φ \\ -sin φ \end{bmatrix}
\]

Pictorally, we have

\[
\text{Notice how this forms a natural coordinate system relative to the surface of the sphere. This is very convenient in a number of cases such as, for example, dynamics on the surface of the Earth.}
\]

(Side note: notice how we have

\[
\frac{∂r}{∂ρ} \frac{∂r}{∂θ} \frac{∂r}{∂φ} = ρ \quad \frac{∂r}{∂ρ} \frac{∂r}{∂θ} \frac{∂r}{∂φ} = ρ^2 sin φ
\]

Recall that we had the relationship \( dv = (ρ) dρ dθ dz \). In spherical coordinates, we have \( (ρ^2 sin φ) dρ dθ dφ \). This multiplier is called the Jacobian, and it pops up frequently when working with curvilinear coordinates.)
Let us now consider the derivatives of the unit vectors in spherical coordinates. We can see straight off that all of the derivatives with respect to $\rho$ will be zero, but the other ones will be nontrivial.

\[
\frac{\partial \mathbf{g}_\rho}{\partial \theta} = \begin{bmatrix}
-\sin \theta \sin \phi \\
\cos \theta \sin \phi \\
0
\end{bmatrix} = \sin \phi \mathbf{g}_\theta
\]

\[
\frac{\partial \mathbf{g}_\theta}{\partial \theta} = \begin{bmatrix}
-\cos \theta \\
-\sin \theta \\
0
\end{bmatrix} = \sin \phi \begin{bmatrix}
-\cos \theta \sin \phi \\
-\sin \theta \sin \phi \\
-\cos \phi
\end{bmatrix} + \cos \phi \begin{bmatrix}
-\cos \theta \cos \phi \\
-\sin \theta \cos \phi \\
\sin \phi
\end{bmatrix}
\]

\[
\frac{\partial \mathbf{g}_\theta}{\partial \phi} = \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix} = 0
\]

\[
\frac{\partial \mathbf{g}_\phi}{\partial \theta} = \begin{bmatrix}
-\sin \theta \cos \phi \\
\cos \theta \cos \phi \\
0
\end{bmatrix} = \cos \phi \mathbf{g}_\theta
\]

\[
\frac{\partial \mathbf{g}_\phi}{\partial \phi} = \begin{bmatrix}
-\cos \theta \sin \phi \\
-\sin \theta \sin \phi \\
-\cos \phi
\end{bmatrix} = -\mathbf{g}_\rho
\]
11.3 Position, velocity, acceleration

Let a particle at time \( t \) be located at

\[
\mathbf{r}(t) = \begin{bmatrix} r_x(t) \\ r_y(t) \\ r_z(t) \end{bmatrix}
\]  

(11.31)

where \( r_x, r_y, r_z \) are the time-dependent coordinates in the \( x, y, \) and \( z \) directions, and \( \mathbf{r} \) is the distance or position vector.

Suppose that the particle is moving at a velocity \( \mathbf{v} \). How far does the particle move over a time period of \( \Delta t \)?

\[
\mathbf{v} \Delta t = \mathbf{r}(t + \Delta t) - \mathbf{r}(t)
\]

(11.32)

or, taking the limit

\[
\mathbf{v}(t) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} = \frac{d}{dt} \mathbf{r}(t) = \begin{bmatrix} \frac{d}{dt} r_x(t) \\ \frac{d}{dt} r_y(t) \\ \frac{d}{dt} r_z(t) \end{bmatrix} = \begin{bmatrix} v_x(t) \\ v_y(t) \\ v_z(t) \end{bmatrix}
\]

(11.33)

What is the direction of the velocity vector?

The velocity direction is always tangent to the line of motion.

What is the magnitude of the velocity vector?

\[
|\mathbf{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2} = s
\]

(11.34)
The magnitude is the speed.
Similarly, the acceleration of the particle is
\[ a(t) = \frac{d}{dt} v(t) = \frac{d^2}{dt^2} r(t) \]  
\[ (11.35) \]

What is the direction of the acceleration vector? That’s a little trickier: there are two types of acceleration.

**Normal acceleration** is the change in velocity direction, **tangential acceleration** is the change in velocity magnitude.

### 11.3.1 Normal and tangential components of acceleration

How can we get the tangential component of acceleration? We know that it will be the amount of acceleration in the same direction as velocity—so we can use the projection of the acceleration on the velocity vector. Therefore the **tangential** component of acceleration is given by

\[ a_{\parallel} = \frac{a \cdot v}{||v||} \]  
\[ (11.36) \]

We can write the vector version of this as

\[ a_{\parallel} = a_{\parallel} t = a_{\parallel} \left( \frac{a \cdot v}{||v||} \right) = \left( \frac{a \cdot v}{v \cdot v} \right) v \]  
\[ (11.37) \]

What about the normal component of acceleration?

We can get this by using vector addition:

\[ a = a_{\parallel} + a_{\perp} \Rightarrow a_{\perp} = a - \left( \frac{a \cdot v}{v \cdot v} \right) v \]  
\[ (11.38) \]

**Example 11.2**

A particle follows the trajectory
\[ r(t) = \left[ t \frac{c_1}{\sqrt{(t c_2)^2 + \ell^2}} \right] \]  
\[ (11.39) \]

where \( c, b \) are in units of \([\text{length}] / [\text{time}]\) and \( \ell \) is in units of \([\text{length}]\). Compute the velocity, acceleration, and normal/tangential components of acceleration.

The velocity and acceleration are obtained by differentiation:

\[ v(t) = \frac{d}{dt} r(t) = \left[ \frac{c_1}{\ell^2/c_2^2} \right] \]  
\[ a(t) = \left[ \frac{\ell/c_2 (t^2 + \ell^2/c_2^2)^{3/2}}{0} \right] \]  
\[ (11.40) \]
The tangential component is given by
\[
a_{\parallel} = \left( \frac{a \cdot v}{v \cdot v} \right) v = \frac{\ell t/c_2(t^2 + \ell^2/c_2^2)^{3/2}}{c_1^2 + t/(t^2 + \ell^2/c_2^2)} \begin{bmatrix} \frac{c_1}{0} \\ \frac{\sqrt{t^2 + \ell^2/c_2^2}}{0} \end{bmatrix}
\]  
(11.41)

and the normal is given by
\[
a_{\perp} = \left[ \frac{\ell}{c_2(t^2 + \ell^2/c_2^2)^{3/2}} - \frac{\ell t/c_2((0)^2 + \ell^2/c_2^2)^{3/2}}{c_1^2 + t/(t^2 + \ell^2/c_2^2)} \right] \begin{bmatrix} 0 \\ \frac{\sqrt{t^2 + \ell^2/c_2^2}}{0} \end{bmatrix}
\]  
(11.42)

Sanity check: what are the accelerations at \( t = 0 \)?
\[
a_{\parallel}(0) = \frac{\ell (0)/c_2((0)^2 + \ell^2/c_2^2)^{3/2}}{c_1^2 + (0)/(0)^2 + \ell^2/c_2^2)} \begin{bmatrix} 0 \\ \frac{\sqrt{0^2 + \ell^2/c_2^2}}{0} \end{bmatrix} = 0
\]
(11.43)
\[
a_{\perp}(0) = \left[ \frac{\ell}{c_2((0)^2 + \ell^2/c_2^2)^{3/2}} - 0 \right] = \begin{bmatrix} 0 \\ -\frac{\ell^2/c_2^2}{0} \end{bmatrix}
\]  
(11.44)

Does this make sense? Yes: the velocity is not changing here, only the direction. What about units? We have
\[
\frac{[\text{length}]^2/[\text{time}]^2}{[\text{length}]^2} = \frac{[\text{length}]^2}{[\text{time}]^2} = [\text{acceleration}] \quad \checkmark
\]  
(11.45)

### 11.4 Natural basis

We note that \( v \) is orthogonal to \( a_{\perp} \). Occasionally, it is useful to construct a coordinate system from these vectors:
\[
\tau = \frac{1}{||v||} v \quad \quad \quad n = \frac{1}{||a_{\perp}||} a_{\perp}
\]  
(11.46)

We need a third unit vector to form a complete basis—where can we get it? We can use the cross product:
\[
b \triangleq \tau \times n
\]  
(11.47)

Check a couple of properties: what is the magnitude of \( b \)?
\[
||b|| = ||\tau \times n|| = ||\tau|| ||n|| \sin 90^\circ = 1
\]  
(11.48)

Is \( b \) orthogonal to \( \tau \) and \( n \)? Yes—crossing two vectors yields a vector that is mutually orthogonal. Therefore, \( (\tau, n, b) \) form an orthonormal basis.

What happens when the particle is traveling in a straight line? (i.e. the normal acceleration is zero) Then \( n \) and \( b \) are ill-defined.

### 11.5 Partial and total derivatives

As we have seen, we frequently encounter situations in which we are working with multiple variables, meaning that we have to be careful about how we take derivatives.

Suppose that we have the following multivariate function
\[
f(x, y, z).
\]  
(11.49)
What does it mean to differentiate this function? Introducing multiple variables results in multiple possible methods of differentiation. The simplest is to take a partial derivative with respect to one of the arguments: here, we can write

\[
\frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial y} \quad \frac{\partial f}{\partial z}
\]

(11.50)

Partial differentiation is generally pretty easy: we just pretend that the other arguments are constant. Now, suppose that \( x, y, z \) are all functions of some other variable, e.g. \( x(t), y(t), z(t) \), so that we have

\[
f(x(t), y(t), z(t))
\]

(11.51)

What is the derivative of \( f \) with respect to \( t \)? We use the chain rule here to write

\[
\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial z} \frac{dz}{dt}
\]

(11.52)

This is called the total derivative.

What if \( x \) was actually a function of two variables, e.g. \( x(t, s) \) Then we have to compute the total derivative of \( x \):

\[
\frac{dx}{dt} = \frac{\partial x}{\partial t} \frac{dt}{dt} + \frac{\partial x}{\partial s} \frac{ds}{dt}
\]

(11.53)

Notice that if \( s \) does not depend on \( t \), then the total derivative of \( x \) w.r.t. \( t \) is equal to the partial derivative.

### 11.5.1 Total derivatives of unit vectors

In the last lecture, we talked a lot about finding the partial derivatives of unit vectors. In Cartesian coordinates, these are pretty boring— they are just constants. In curvilinear coordinates, however, they are a lot more interesting. Suppose we have our usual cylindrical unit vectors \( \hat{g}_\rho(\rho, \theta, z), \hat{g}_\theta(\rho, \theta, z), \hat{g}_z(\rho, \theta, z) \). Additionally, suppose that the coordinates are functions of time, i.e. \( \rho(t), \theta(t), z(t) \). What are the total derivatives with respect to time?

\[
\frac{d}{dt} \hat{g}_\rho(\rho(t), \theta(t), z(t)) = \frac{\partial \hat{g}_\rho}{\partial \rho} \frac{d\rho}{dt} + \frac{\partial \hat{g}_\rho}{\partial \theta} \frac{d\theta}{dt} + \frac{\partial \hat{g}_\rho}{\partial z} \frac{dz}{dt} = \hat{g}_\theta \frac{d\theta}{dt}
\]

(11.54)

\[
\frac{d}{dt} \hat{g}_\theta(\rho(t), \theta(t), z(t)) = \frac{\partial \hat{g}_\theta}{\partial \rho} \frac{d\rho}{dt} + \frac{\partial \hat{g}_\theta}{\partial \theta} \frac{d\theta}{dt} + \frac{\partial \hat{g}_\theta}{\partial z} \frac{dz}{dt} = -\hat{g}_\rho \frac{d\rho}{dt}
\]

(11.55)

\[
\frac{d}{dt} \hat{g}_z(\rho(t), \theta(t), z(t)) = \frac{\partial \hat{g}_z}{\partial \rho} \frac{d\rho}{dt} + \frac{\partial \hat{g}_z}{\partial \theta} \frac{d\theta}{dt} + \frac{\partial \hat{g}_z}{\partial z} \frac{dz}{dt} = 0
\]

(11.56)

For reference, let us compute the same thing for spherical coordinates:

\[
\frac{d}{dt} \hat{g}_\rho(\rho(t), \theta(t), \phi(t)) = \frac{\partial \hat{g}_\rho}{\partial \rho} \frac{d\rho}{dt} + \frac{\partial \hat{g}_\rho}{\partial \theta} \frac{d\theta}{dt} + \frac{\partial \hat{g}_\rho}{\partial \phi} \frac{d\phi}{dt} = \hat{g}_\rho \sin \phi \frac{d\theta}{dt} + \hat{g}_\phi \frac{d\phi}{dt}
\]

(11.57)

\[
\frac{d}{dt} \hat{g}_\theta(\rho(t), \theta(t), \phi(t)) = \frac{\partial \hat{g}_\theta}{\partial \rho} \frac{d\rho}{dt} + \frac{\partial \hat{g}_\theta}{\partial \theta} \frac{d\theta}{dt} + \frac{\partial \hat{g}_\theta}{\partial \phi} \frac{d\phi}{dt} = -\hat{g}_\rho \sin \phi \frac{d\rho}{dt} - \hat{g}_\phi \cos \phi \frac{d\phi}{dt}
\]

(11.58)

\[
\frac{d}{dt} \hat{g}_\phi(\rho(t), \theta(t), \phi(t)) = \frac{\partial \hat{g}_\phi}{\partial \rho} \frac{d\rho}{dt} + \frac{\partial \hat{g}_\phi}{\partial \theta} \frac{d\theta}{dt} + \frac{\partial \hat{g}_\phi}{\partial \phi} \frac{d\phi}{dt} = \hat{g}_\theta \cos \phi \frac{d\theta}{dt} - \hat{g}_\rho \frac{d\phi}{dt}
\]

(11.59)

As you can see, it is fairly tedious to compute these quantities. However, it will make the derivation of future quantities much easier.
11.6  Velocity and acceleration in curvilinear coordinates

Let us suppose that a particle moves along a path. The coordinates of the particle are given as a function of time.

\[ r = r(x(t), y(t), z(t)) \]  \hspace{1cm} (11.60)

We compute the velocity by taking the total derivative of position with respect to time. In Cartesian coordinates, this comes out to be

\[ v = \frac{d}{dt} r(x(t), y(t), z(t)) = \frac{\partial r}{\partial x} \frac{dx}{dt} + \frac{\partial r}{\partial y} \frac{dy}{dt} + \frac{\partial r}{\partial z} \frac{dz}{dt} = \frac{dx}{dt} \mathbf{g}_x + \frac{dy}{dt} \mathbf{g}_y + \frac{dz}{dt} \mathbf{g}_z \]  \hspace{1cm} (11.61)

Similarly, we take the total derivative of velocity to obtain the acceleration. Notice that this is very easy to compute, as the unit vectors are constant.

\[ a = \frac{dv}{dt} = \frac{d^2 x}{dt^2} \mathbf{g}_x + \frac{d^2 y}{dt^2} \mathbf{g}_y + \frac{d^2 z}{dt^2} \mathbf{g}_z \]  \hspace{1cm} (11.62)

Easy enough. Now, let us suppose that we describe the position of the particle using cylindrical coordinates, so that the position is given as:

\[ r = r(\rho(t), \theta(t), z(t)) \]  \hspace{1cm} (11.63)

Now, we will follow exactly the same procedure.

\[ v = \frac{d}{dt} r(\rho(t), \theta(t), z(t)) = \frac{\partial r}{\partial \rho} \frac{d\rho}{dt} + \frac{\partial r}{\partial \theta} \frac{d\theta}{dt} + \frac{\partial r}{\partial z} \frac{dz}{dt} = \frac{d\rho}{dt} \mathbf{g}_\rho + \frac{d\theta}{dt} \mathbf{g}_\theta + \frac{dz}{dt} \mathbf{g}_z \]  \hspace{1cm} (11.64)

Note that we have a stray \( \rho \) floating around. This is an artifact of the coordinate system we are using. Now, let us compute the acceleration. As before, we simply take the total derivative with respect to time. However, things are going to immediately get more complicated, because the derivatives of the unit vectors will not be zero.

\[ a = \frac{dv}{dt} = \frac{d^2 \rho}{dt^2} \mathbf{g}_\rho + \frac{d^2 \theta}{dt^2} \mathbf{g}_\theta + \frac{dz}{dt} \mathbf{g}_z \]  \hspace{1cm} (11.65)

Differentiate termwise...

\[ a = \frac{d^2 \rho}{dt^2} \mathbf{g}_\rho + \frac{d^2 \rho}{dt^2} \mathbf{g}_\theta + \frac{d^2 \theta}{dt^2} \mathbf{g}_\theta + \frac{dz}{dt} \mathbf{g}_z \]  \hspace{1cm} (11.66)

Gathering terms...

\[ = \mathbf{g}_\rho \left[ \frac{d^2 \rho}{dt^2} - \rho \left( \frac{d\theta}{dt} \right)^2 \right] + \mathbf{g}_\theta \left[ 2 \frac{d\theta}{dt} \frac{d\rho}{dt} + \frac{d^2 \theta}{dt^2} \right] + \mathbf{g}_z \left[ \frac{dz}{dt} \right] \]  \hspace{1cm} (11.67)

Derivatives are frequently denoted with dots to make the final expression easier:

\[ = \mathbf{g}_\rho (\ddot{\rho} - \rho(\dot{\theta})^2) + \mathbf{g}_\theta (2\dot{\theta} \dot{\rho} + \ddot{\theta}) + \mathbf{g}_z (\ddot{z}) \]  \hspace{1cm} (11.69)

Finally, let’s do the same thing for spherical coordinates.

\[ r = r(\rho(t), \theta(t), \phi(t)) \]  \hspace{1cm} (11.70)
Taking the total derivative...

\[
\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{\partial \mathbf{r}}{\partial \rho} \frac{d\rho}{dt} + \frac{\partial \mathbf{r}}{\partial \theta} \frac{d\theta}{dt} + \frac{\partial \mathbf{r}}{\partial \phi} \frac{d\phi}{dt} = \mathbf{g}_\rho \frac{d\rho}{dt} + \mathbf{g}_\theta \rho \sin \phi \frac{d\theta}{dt} + \mathbf{g}_\phi \rho \frac{d\phi}{dt} \tag{11.71}
\]

And finally, for the acceleration we have

\[
\mathbf{a} = \frac{d^2 \mathbf{r}}{dt^2} = \frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial \rho} \frac{d\rho}{dt} + \frac{\partial \mathbf{v}}{\partial \theta} \frac{d\theta}{dt} + \frac{\partial \mathbf{v}}{\partial \phi} \frac{d\phi}{dt} = \mathbf{g}_\rho \frac{d^2 \rho}{dt^2} + \mathbf{g}_\theta \rho \sin \phi \frac{d^2 \theta}{dt^2} + \mathbf{g}_\phi \rho \frac{d^2 \phi}{dt^2} \tag{11.72}
\]

\[
= \left[ g_\rho \dot{\phi} \sin \phi + g_\phi \dot{\rho} \right] \ddot{\rho} + g_\rho \dot{\rho} \ddot{\rho} + g_\phi \dot{\phi} \ddot{\phi} + g_\phi \rho \ddot{\phi} + g_\phi \rho \dot{\phi} + g_\phi \rho \ddot{\phi} \tag{11.73}
\]

\[
\left[ -g_\rho \dot{\rho} \sin \phi - g_\phi \dot{\phi} \cos \phi \right] \rho (\sin \phi) \dot{\theta} + g_\phi \rho \left[ \dot{\phi} \sin \phi + \rho \dot{\phi} \cos \phi \right] + g_\phi \rho (\sin \phi) \ddot{\theta} + \left[ g_\phi \rho \dot{\phi} \sin \phi + g_\phi \rho \dot{\phi} \cos \phi \right] + g_\phi \rho \left( 2 \dot{\phi} \sin \phi + 2 \rho \dot{\phi} \cos \phi + \rho \ddot{\phi} \sin \phi \right) \tag{11.74}
\]

\[
= g_\rho \left[ \rho - \rho \ddot{\phi} \sin^2 \phi - \rho \dot{\phi}^2 \right] + g_\phi \left[ 2 \dot{\phi} \sin \phi + 2 \rho \dot{\phi} \cos \phi + \rho \ddot{\phi} \sin \phi \right] + g_\phi \left[ 2 \dot{\phi} \sin \phi + \rho \ddot{\phi} \sin \phi \right] \tag{11.75}
\]

which, as you can see, is horribly tedious. Fortunately, now that we’ve derived these equations, we can simply refer to this formula in the future.

**Example 11.3**

Consider a ball moving in a circle of radius \( R \) at a rate of \( \Omega \) radians per second.

Find the velocity and acceleration of the ball. Here we can simply use our formula. We have

\[
\rho(t) = R, \quad \theta(t) = \Omega t, \quad z(t) = 0 \tag{11.77}
\]

Now, let us substitute this into our expression for velocity:

\[
\mathbf{v} = g_\rho \frac{d^2 \rho}{dt^2} + g_\theta \rho \frac{d^2 \theta}{dt^2} + g_\phi \frac{d^2 \phi}{dt^2} = g_\rho R \Omega \tag{11.78}
\]

Does this make sense? We note that \( R \Omega \) is the *angular velocity*, indicating that the magnitude is correct. The direction is correct as well—always tangent to the circle. Now, let’s compute the acceleration:

\[
\mathbf{a} = g_\rho \left[ \frac{d^2 \rho}{dt^2} - \rho \left( \frac{d \theta}{dt} \right)^2 \right] + g_\theta \left[ 2 \frac{d \theta}{dt} \frac{d^2 \rho}{dt^2} + \rho \frac{d^2 \phi}{dt^2} \right] + g_\phi \left[ \frac{d^2 \phi}{dt^2} \right] = -g_\rho R \Omega^2 \tag{11.79}
\]

We note that the acceleration is pointing directly inwards, which is correct.
Example 11.4

Santa Claus leaves the North Pole and heads directly for the equator. The Earth has a radius of $R$, and he travels at a rate of $v = R\omega$, where $\omega$ is his angular velocity in radians per second.

In addition to this motion, the Earth is rotating at a rate of $\Omega$ radians per second. In spherical coordinates, then, his position as a function of time is given by

$$
\rho(t) = R \quad \theta(t) = \Omega t \quad \phi(t) = \omega t
$$

What is his velocity and acceleration in spherical coordinates?

We can use our formulae here. For velocity we have:

$$
\mathbf{v}(t) = \mathbf{g}_\rho \frac{d\rho}{dt} + \mathbf{g}_\theta \rho \sin \phi \frac{d\theta}{dt} + \mathbf{g}_\phi \rho \frac{d\phi}{dt} = g_\theta R \Omega \sin(\omega t) + g_\phi R \omega
$$

Let's consider these terms individually. We have no radial terms, as we expect, so the remaining velocity is tangent to the surface. We see that the $\theta$ velocity is maximized at the equator and is zero at the North Pole. On the other hand, the $\phi$ velocity is constant. So far, so good. Now, let's consider the acceleration.
For acceleration:

\[ a = g_\rho \left[ \dot{\rho}^2 (\rho \phi^2 - \rho \dot{\phi}^2) \right] + g_\rho \left[ 2 \rho \dot{\rho} \sin \phi + 2 \rho \dot{\phi} \cos \phi + \rho \dot{\phi} \sin \phi \right] + g_\rho \left[ 2 \rho \phi^2 + \rho \dot{\phi}^2 - \rho \dot{\rho}^2 \sin \phi \cos \phi \right] \]

\[ = -g_\rho \left[ R \Omega^2 \sin^2(\omega t) + R \omega^2 \right] + g_\rho \left[ 2 R \Omega \omega \cos(\omega t) \right] - g_\rho \left[ R \Omega^2 \sin(\omega t) \cos(\omega t) \right] \]

Let's analyze each of the terms in the result. The normal acceleration terms may seem fairly straightforward, as they are the result of rotational motion. As expected, we have a contribution from both rotational terms. The \( g_\phi \) term is the centripetal acceleration. Remember, the point is accelerating inwards, so there will be a \( \phi \) component. Note for what values of \( \Omega, \omega \), the term is zero.

What about the \( g_\theta \) term? This is called the Coriolis acceleration term. It is the result of traveling on a spherical rotating surface, and it is a real effect that must be taken into account during air travel. When is it maximized and minimized? When is it positive and negative?

In addition to air travel, the Coriolis effect affects a number of other things such as the pendulum in the foyer of OCSE, and the direction in which hurricanes travel.

It is worth noting that the Coriolis effect has nothing to do with which way water drains. This is, in fact, an urban legend – the effects of momentum and viscosity are much more dominant on that scale.

### 11.7 Angular velocity

Consider a particle moving at a velocity \( \mathbf{v} \) that is moving in a circular motion as shown in the following figure:

Let us select a point of reference that is the center of the rotation of the particle, so that the position vector is \( \mathbf{r} \) as shown. We note that \( \mathbf{r} \) and \( \mathbf{v} \) are orthogonal, so that \( \mathbf{r} \cdot \mathbf{v} = 0 \).

We define the angular velocity as follows:

\[ \boldsymbol{\omega} \triangleq \frac{\mathbf{r} \times \mathbf{v}}{||\mathbf{r}||^2} \]

Note that the angular velocity is orthogonal to the plane in which rotation takes place. The angular velocity vector is similar to the moment vector that we are familiar with.

What is the magnitude of the angular velocity vector? We can compute this as

\[ ||\boldsymbol{\omega}|| = \left| \frac{\mathbf{r} \times \mathbf{v}}{||\mathbf{r}||} \right| = \frac{1}{||\mathbf{r}||} ||\mathbf{v}|| \sin \theta = \frac{||\mathbf{v}||}{||\mathbf{r}||} \]

or, letting \( ||\mathbf{v}|| = \omega, ||\mathbf{r}|| = r, ||\mathbf{v}|| = v \) and rearranging, we have

\[ v = r \omega \]

which is what we are familiar with from elementary physics.

We have \( \omega \) in terms of \( \mathbf{r}, \mathbf{v} \), but it is also useful to have \( \mathbf{v} \) in terms of \( \mathbf{r}, \omega \). Let's attempt to isolate (11.84) for \( \mathbf{v} \). To do this, we will need an identity: let \( \mathbf{a}, \mathbf{b}, \mathbf{c} \) be vectors. Then

\[ \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}) \]

You can go through and prove this yourself if you like.
Now, let’s apply this to (11.84) and cross both sides by \( r \) on the left.

\[
\omega (r \cdot r) = r \times v
\]  
(11.88)

\[
(r \times \omega)(r \cdot r) = r \times (r \times v)
\]  
(11.89)

Now we can use our identity:

\[
(r \times \omega)(r \cdot r) = r(r \cdot v) - v(r \cdot r)
\]  
(11.90)

Here we use the fact that \( r \) and \( v \) are orthogonal to obtain

\[
(r \times \omega)(r \cdot r) = -v(r \cdot r)
\]  
(11.91)

\[
r \times \omega = -v
\]  
(11.92)

\[
v = \omega \times r
\]  
(11.93)

which gives us a nice expression for \( v \) in terms of \( r \) and \( \omega \).

### 11.8 Relative Motion

Sometimes it is useful to compute the motion of one moving particle with respect to another moving particle, as in the following scenario.

\[
\begin{align*}
\dot{r}_{ab}(t) & = \dot{r}_a(t) - \dot{r}_b(t) \\
\ddot{r}_{ab}(t) & = \ddot{r}_a(t) - \ddot{r}_b(t)
\end{align*}
\]

The relative position of the particle is

\[
r_{ab}(t) = r_a(t) - r_b(t)
\]  
(11.94)

By linearity of the derivative operator, we differentiate both sides to obtain

\[
\begin{align*}
v_{ab}(t) & = v_a(t) - v_b(t) \\
\ddot{v}_{ab}(t) & = \dddot{v}_a(t) - \dddot{v}_b(t) \\
a_{ab}(t) & = a_a(t) - a_b(t)
\end{align*}
\]  
(11.95)

(11.96)

### Example 11.5

A wheel of radius \( R \) is rolling along a surface at a velocity \( v_0 \).

Find the velocity vector for the point on the wheel that is in contact with the ground at time \( t = 0 \). We use relative motion and our angular velocity technology to solve this problem.
We can break the total velocity up into two components:

\[
\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2
\]

The translational velocity is given by

\[
\mathbf{v}_1 = \begin{bmatrix} v_0 \\ 0 \\ 0 \end{bmatrix}
\]

(11.97)

The rotational velocity is given by

\[
\mathbf{v}_2 = \mathbf{r}_2 \times \omega
\]

(11.98)

Let us assume that the wheel is turning at a rate \(\omega\) (radians/second). Then we have

\[
\omega = \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix}
\]

\[
\mathbf{r}_2 = \begin{bmatrix} -R \sin(\omega t) \\ -R \cos(\omega t) \\ 0 \end{bmatrix}
\]

(11.100)

so the rotational velocity is

\[
\mathbf{v}_2 = \begin{bmatrix} \hat{i} \\ -R \sin(\omega t) \\ -R \cos(\omega t) \\ 0 \end{bmatrix}
\]

(11.101)

and the total velocity is

\[
\mathbf{v} = \begin{bmatrix} v_0 - \omega R \cos \omega \\ \omega R \sin \omega \\ \omega \end{bmatrix}
\]

(11.102)

We still don't know what \(\omega\) is, so let's solve for it. We are looking at the particle that is in contact with the ground at \(t = 0\). Therefore, what should its velocity be? It should be zero, because the ground is not moving. Therefore we have

\[
\mathbf{v}(0) = \begin{bmatrix} v_0 - \omega R \\ \omega R \sin \omega \\ 0 \end{bmatrix} = 0 \implies \omega = \frac{v_0}{R}
\]

(11.103)

so we obtain the result

\[
\mathbf{v} = \begin{bmatrix} v_0(1 - \cos \omega) \\ \omega R \sin \omega \\ 0 \end{bmatrix}
\]

(11.104)

This concludes the content of MAE2103. The concepts introduced here will be continued in MAE2104, Engineering Mechanics II. They will also carry over into MAE3201, Strength of Materials.
Midterm Review

In the first half of the course, we have developed the mathematical structure for formulating the equations of statics. We began by introducing forces, the principle of force equilibrium, and applications of force equilibrium to single-particle systems. We then introduced moments, the principle of moment equilibrium, and applications of moment equilibrium to rigid-body systems. From there, we applied these concepts to other specialized systems such as trusses and multi-rigid-body systems. Finally, we introduced internal forces and showed how the equations of equilibrium are used to find them.

A.1 Linear Algebra

- **2D/3D vectors:** we represent distances, forces, and moments as vectors.
  
  - The **magnitude** of a vector \( \mathbf{f} \) is \( ||\mathbf{f}|| = \sqrt{f_x^2 + f_y^2 + f_z^2} \)
  
  - **Unit vectors** are unitless vectors with magnitude 1.
  
  - Any vector can be represented as \( \mathbf{f} = ||\mathbf{f}|| \mathbf{n} \) (A.1)

  where \( ||\mathbf{f}|| \) is the magnitude of the vector and \( \mathbf{n} \) is a unit vector pointing in the same direction.

- **Vector products:** we introduced two key important vector products.

  - The **dot product** is \( \mathbf{f} \cdot \mathbf{g} = f_x g_x + f_y g_y + f_z g_z \).
    
    \( \implies \) We also know that \( \mathbf{f} \cdot \mathbf{g} = ||\mathbf{f}|| ||\mathbf{g}|| \cos \theta \) where \( \theta \) is the angle between them.
    
    \( \implies \) \( \mathbf{f} \cdot \mathbf{g} = 0 \iff \mathbf{f} \parallel \mathbf{g} \) orthogonal.

  - The **cross product** is given by the mnemonic

    \[ \mathbf{f} \times \mathbf{g} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ f_x & f_y & f_z \\ g_x & g_y & g_z \end{bmatrix} \] (A.2)

    \( \implies \) We also know that \( ||\mathbf{f} \times \mathbf{g}|| = ||\mathbf{f}|| ||\mathbf{g}|| \sin \theta \)
    
    \( \implies \) In 2D, cross product will only have a \( z \) component, which will be equal to \( ||\mathbf{f}|| ||\mathbf{g}|| \sin \theta \) (figure the sign out using right hand rule)
    
    \( \implies \) \( \mathbf{f} \times \mathbf{g} = 0 \iff \mathbf{f} \perp \mathbf{g} \) parallel.

- **Vector equations:** we write our force and moment equilibrium equations in vector form.

  \[ \mathbf{f} + \mathbf{g} + \mathbf{h} = 0 \implies \begin{bmatrix} f_x + g_x + h_x \\ f_y + g_y + h_y \\ f_z + g_z + h_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies \begin{cases} f_x + g_x + h_x = 0 \\ f_y + g_y + h_y = 0 \\ f_z + g_z + h_z = 0 \end{cases} \] (A.3)

  In 2D a vector equation is 2 equations, in 3D it is 3 equations.

- **Linear systems:** we frequently need to solve 2x2 or 3x3 systems of equations where we cannot solve for any variable easily. Systems like these can be written in matrix form:

  \[ \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \] (A.4)

  where \( a_{11} \ldots a_{33} \) and \( b_1 \ldots b_3 \) are known and \( x_1 \ldots x_3 \) are unknown. Systems like this can be solved using substitution, Gauss elimination, or Cramer’s rule.
A.2 Force Equilibrium

- Equations of force equilibrium:

\[
\sum f = f_1 + f_2 + \ldots = \begin{bmatrix} f_{1x} + f_{2x} + \ldots \\ f_{1y} + f_{2y} + \ldots \\ f_{1z} + f_{2z} + \ldots \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\]  

(A.5)

In 2D there are 2 equations of force equilibrium, in 3D there are 3.

- Two force members: a member is a two force member if and only if it has only two forces (and no moments) applied—generally one at each end.
  - Forces are equal and opposite
  - Forces are colinear with the beam: that is,

\[
f = ||f||n
\]

(A.6)

where \( n \) is a unit vector parallel to the member and \( ||f|| \) is the magnitude of the force, which we generally call the tension.

- Example: compute the force vectors for the following cables:

First, we find the unit vectors for each cable. We can do this by finding the vector that points from one end of the cable to the other, and then divide by the magnitude:

\[
n_1 = \begin{bmatrix} -6m \\ 3m \\ 2m \end{bmatrix} \times \frac{1}{\sqrt{(6m)^2 + (3m)^2 + (2m)^2}} = \begin{bmatrix} -6/7 \\ 3/7 \\ 2/7 \end{bmatrix}
\]

(A.7)

\[
n_2 = \begin{bmatrix} -6m \\ 2m \\ 3m \end{bmatrix} \times \frac{1}{7m} = \begin{bmatrix} -6/7 \\ 2/7 \\ 3/7 \end{bmatrix}
\]

(A.8)

\[
n_3 = \begin{bmatrix} 1m \\ 0m \\ 0m \end{bmatrix} \times \frac{1}{\sqrt{(1m)^2}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}
\]

(A.9)

We get the force vectors by multiplying the unit vectors by the tensions.
A.3 Moments

- Definition of moments: the moment of a force about a point is \( M = r \times f \) where \( r \) is a distance vector pointing from the reference point to the point of load application:

\[
M = r \times f
\]

- The above is the moment of \( f_b \) about point \( a \).
- Distance vectors have units of length, force vectors have units of force.

- Line of action: a force's moment will always be the same if you translate it along its line of action, as shown:

One way to compute a moment (in 2D) is to translate the force so that it is orthogonal to its distance vector—above this is \( r_\perp \). Then, the moment is

\[
M_z = \pm ||r_\perp||f|| \sin \theta = ||f_\perp||f||
\]

- Computing moment of force without cross product: in 2D, we only need to find the \( z \) component. Consider the following example where we compute the moment of \( f \) about \( a \).

We can compute the magnitude simply by translating the force \( f \) up so that it is a distance \( L \) from \( a \). Using the right hand rule, we have that \( M_z = ||f||L \).

- Couple moments: a couple moment is the result of two equal and opposite forces that are displaced from each other in order to create pure moment with no resultant force.
Irrelevant for force balance, but must be accounted for in moment balance.
Location is generally irrelevant for rigid body analysis.

Computing effective force and moment at a specific location: we can replace any force/moment system by a single force and couple moment acting at a point—we can call these "effective forces" and "effective moments."

- The effective force is just the sum of all the forces.
- The effective moment is just the sum of the moments about the point.

So in the above example we could translate the force to \( a \) as long as we take its effective moment with it:

\[
\begin{align*}
\text{f}_{\text{eff}} &= f \\
\text{M}_{\text{eff}} &= \begin{bmatrix} 0 \\ 0 \\ fL \end{bmatrix}
\end{align*}
\]

- Force/moment of a distributed load
  - The effective force of a distributed load \( w(x) \) is
    \[
    \int_{a}^{b} w(x) \, dx
    \]
  - The effective moment is
    \[
    \int_{a}^{b} w(x) r(x) \, dx
    \]
    where \( x \) is an integration variable and \( r(x) \) is the distance from the reference point to \( x \). (Always be careful with your signs and make sure to check your results against the right hand rule.)

\[\text{(A.10)}\]

\[\text{(A.11)}\]

### A.4 Rigid Body Equilibrium

- Force and moment equilibrium: in addition to force equilibrium, rigid bodies satisfy moment equilibrium:
  \[
  \sum \text{M} = 0
  \]
  about any and all reference points, in or outside the rigid body.

- Reaction forces
  - Supports show up as reaction forces in a free body diagram.
  - Constraint restricts motion in all directions \( \Leftrightarrow \) (e.g. a pivot)
    \[
    \text{f}_{\text{reaction}} = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix}
    \]
  - Free motion in all directions = zero reaction force (e.g. free end)
- Constraint restricts motion in one direction only:

\[ \mathbf{f}_{\text{reaction}} = |\mathbf{f}_{\text{reaction}}| \mathbf{n} \quad (A.14) \]

where \( \mathbf{n} \) is the direction that the body is not allowed to move.
- Constraint restricts rotation in all directions (e.g. a fixed joint)

\[ \mathbf{M}_{\text{reaction}} = \begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} \quad (A.15) \]

and so on. Remember: reaction forces/moments are applied only where position/rotation is constrained.

- Multibody equilibrium and connecting forces:
  - Two rigid bodies calls for two free body diagrams and two sets of equilibrium equations.
  - Represent the effect of the pinned joint by representing them as equal and opposite on the two connected bodies:

\[
\begin{align*}
\mathbf{f}_{\text{pin}} &= \begin{bmatrix} f_{\text{pinx}} \\ f_{\text{piny}} \\ 0 \end{bmatrix} \\
\mathbf{-f}_{\text{pin}} &= \begin{bmatrix} -f_{\text{pinx}} \\ -f_{\text{piny}} \\ 0 \end{bmatrix}
\end{align*}
\]

- For pinned joints have connecting forces; for fixed joints have connecting moments as well.

### A.5 Trusses

- Definition of a truss: a **truss** is a structure constructed only of two-force members.
- Method of joints:
  - Use when you need to analyze **an entire truss**.
  - Procedure:
    1. Determine reaction forces using rigid body analysis (when possible)
    2. Free body diagram for each node
    3. Solve force equilibrium for each node
- Method of sections
  - Use when analyzing **only a couple of members** in a truss.
  - Procedure:
    1. Determine reaction forces (when possible)
    2. Cleverly select a section cutting beams whose tensions you want to compute
    3. Cleverly do force/moment equilibrium

#### Example A.1: Parker Truss
Example: find the forces in members $bc$, $bg$, $hg$:

First question: do you see the zero force members? Members $bh$ and $df$ must have zero tension.

Find reactions $\rightarrow$ method of sections $\rightarrow$ cut as shown $\rightarrow$ take moment about $b$ to get $t_{hg}$.
Final Review part I

B Final Review

B.1 Rigid body analysis

• Force equilibrium:
  \[ \sum f_i = 0 \]
  where \( f_i \) are all of the forces acting on a single object. This is true for every (i) particle, (ii) rigid body, (iii) joint

• Moment equilibrium:
  \[ \sum (r_i - r_0) \times f_i = \sum M_i = 0 \]
  where each \( r_i \) is the point of application of each force \( f_i \), and \( r_0 \) is any point. This is true for every rigid body, and trivially true for other simpler objects.

• Couple moments: If a pair of equal and opposite forces \( f \) (magnitude \( f \)) act on a body and are offset by a distance \( d \) (magnitude \( d \)), then they are said to be a couple moment. Couple moments have the following properties
  - Independent of location
  - Magnitude \( M = fd \)

• Effective forces and moments: For any set of forces \( \{f_i\} \) acting at locations \( r_i \), we say that the effective force and moment of the forces at a point \( r_0 \) is given by
  \[ f_{\text{eff}} = \sum f_i \quad \text{and} \quad M_{\text{eff}} = \sum (r - r_0) \times f_i \]
  Intuitively, \( f_{\text{eff}} \) and \( M_{\text{eff}} \) are the point loads and couple moments that, if applied at \( r_0 \), would cause the system to be in equilibrium.

• Distributed loads: Consider a load distributed along a line according to the function \( w(x) \) in terms of force per unit length. Then the total force is given by
  \[ f_{\text{tot}} = \int_a^b df = \int_a^b w(x) dx \]
  The moment about a point \( r_0 \) is
  \[ M_{\text{tot}} = \int_a^b (r(x) - r_0) \times dF = \int_a^b (r(x) - r_0) \times w_0(x) dx \]
  Generally, the analysis is one-dimensional so everything can be done in terms of magnitudes:
  \[ f_{\text{tot}} = \int_a^b w(x) dx \quad \text{and} \quad M_{\text{tot}} = \int_a^b x w(x) dx \]

  The effective location of a distributed load is given by
  \[ r_{\text{eff}} = \frac{M_{\text{tot}}}{f_{\text{tot}}} \]
Example B.1

Consider the following beam subjected to an applied load.

\[ w(x) = w_0 e^{-x/x_0} \]

The beam is much longer than \( x_0 \) and has negligible weight. Compute the reaction at the fixed joint and the location of the effective force.

To compute the total force we integrate.

\[ f_{\text{eff}} = \int_0^\infty w_0 e^{-x/x_0} \, dx = \left. -w_0 x_0 e^{-x/x_0} \right|_0^\infty = -w_0 x_0 (0 - 1) = w_0 x_0 \]

To compute the total moment we integrate

\[ M_{\text{eff}} = \int_0^\infty w_0 x e^{-x/x_0} \, dx = \left. -w_0 x_0 x e^{-x/x_0} \right|_0^\infty + \int_0^\infty w_0 x_0 e^{-x/x_0} \, dx = -w_0 x_0^2 e^{-x/x_0} \bigg|_0^\infty = w_0 x_0^2 \]

So the location of the effective force is

\[ L_{\text{eff}} = \frac{M_{\text{eff}}}{f_{\text{eff}}} = x_0 \]

• Reaction forces and moments: When doing rigid body analysis, treat each constraint as an unknown applied load that must be solved for. Remember: \textit{constrained motion = unknown applied load}:
  - Roller in the x direction: free to move in x direction, no force in x direction; restricted motion in y direction, applied force in y direction; free to rotate, no applied moment.
  - Pin: Constrained motion in x and y, applied load in x and y; free to rotate, no applied moment.
  - Fixed joint: Fully constrained motion/rotation, fully applied forces/moments.
  - Free end: Fully free motion/rotation, no applied forces/moments.

• Connecting forces and moments and multi-body analysis: Treat each body independently with equal and opposite unknown connecting forces applied.

Example B.2

Consider the following multi-component system
Compute the reactions at the pinned and fixed joints.
We'll follow the recipe here.

1. What are our unknowns?
   - Pinned support: two force reactions $f_{pivx}$, $f_{pivy}$
   - Fixed support: two force reactions $f_{fixx}$, $f_{fixy}$, one moment reaction $M_{fixz}$
   - Slotted pin connection: one force reaction $f_{slot}$
   
   so we have six unknowns.

2. Draw a free body diagram.

3. Write equations:
   Body 1 force equilibrium gives
   $$\sum \mathbf{f} = \begin{bmatrix} f_{pivx} \\ f_{pivy} \\ 0 \end{bmatrix} + \begin{bmatrix} -\sin \theta \\ \cos \theta \\ 0 \end{bmatrix} = \mathbf{0}$$

   Computing moment of the slot about the pivot gives
   $$\mathbf{r} = \begin{bmatrix} W \cos \theta \\ W \sin \theta \\ 0 \end{bmatrix} \implies \mathbf{M} = \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ W f_{slot} \cos \theta^2 + W f_{slot} \sin \theta^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ W f_{slot} \end{bmatrix}$$
so body 1 moment equilibrium gives
\[ \sum \mathbf{M} = \begin{bmatrix} 0 \\ 0 \\ Wf_{\text{slot}} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ M \end{bmatrix} = \mathbf{0} \]

Body 2 force equilibrium gives
\[ \sum \mathbf{f} = f_{\text{slot}} \begin{bmatrix} \sin \theta \\ -\cos \theta \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ f_{\text{fix}} \end{bmatrix} = \mathbf{0} \]

Sum the moments of \( f_{\text{slot}}, \mathbf{f} \) about the fixed joint to get:
\[ \sum \mathbf{M} = \begin{bmatrix} 0 \\ 0 \\ Lf_{\text{slot}} \cos \theta \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2}Lf \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ M \end{bmatrix} \]

(4) Solve the equations:

Body 1 moment z:
\[ Wf_{\text{slot}} + M = 0 \implies f_{\text{slot}} = -\frac{M}{W} \]

Body 1 force x:
\[ f_{\text{pivx}} - f_{\text{slot}} \sin \theta = 0 \implies f_{\text{pivx}} = -\frac{M}{W} \sin \theta \]

Body 1 force y:
\[ f_{\text{pivy}} + f_{\text{slot}} \cos \theta = 0 \implies f_{\text{pivy}} = \frac{M}{W} \cos \theta \]

Body 2 moment z:
\[ Lf_{\text{slot}} \cos \theta + \frac{1}{2}Lf + M = 0 \implies M = \frac{ML}{W} \cos \theta - \frac{1}{2}Lf \]

Body 2 force x:
\[ f_{\text{slot}} \sin \theta + f_{\text{fixx}} = 0 \implies f_{\text{fixx}} = \frac{M}{W} \sin \theta \]

Body 2 force y:
\[ -f_{\text{slot}} \sin \theta - f + f_{\text{fixy}} = 0 \implies f_{\text{fixy}} = f - \frac{M}{W} \sin \theta \]

### B.2 Structural analysis

Trusses are structures that are composed entirely of two-force members with pinned joints. There are two main types of analysis:
• Method of joints: for each node,
\[ \sum t_i n_i = 0 \]  
(B.1)

where \( t_i \) are the tensions in each beam and \( n_i \) are the unit vectors pointing along the beams away from the pinned joint. This forms a system of equations that can be solved.

• Method of sections: select a cut, and perform rigid body analysis.

### B.3 Internal force analysis

• Relationship between applied load, shear, moment:

\[ w(x) = \frac{dV(x)}{dx} \quad V(x) = \frac{dM(x)}{dx} \]

• Boundary conditions: free end \( \implies V = 0, M = 0 \); pinned end \( \implies M = 0, M \neq 0 \), etc.

---

**Example B.3**

A beam is subjected to a distributed load with a triangular shape as shown.

The maximum intensity of the load is \( w_0 \). Find the equations for the shear and moment distributions in the beam.

First we must find the equation for the applied load. We can do this using Heaviside functions:

\[ w(x) = \frac{2w_0}{L} \left( x u(x) + (L - 2x) u \left( x - \frac{L}{2} \right) \right) = \frac{2w_0}{L} \left( x u(x) - 2 \left( x - \frac{L}{2} \right) u \left( x - \frac{L}{2} \right) \right) = \frac{2w_0}{L} \left( r(x) - 2r(x - L/2) \right) \]

We obtain the shear by integration of \( w(x) \):

\[ V(x) = \int w(x) \, dx \]

\[ = \frac{2w_0}{L} \left( \frac{1}{2} x^2 u(x) - \left( x - \frac{L}{2} \right)^2 u \left( x - \frac{L}{2} \right) \right) + C_1 \]

We find \( C_1 \) using boundary conditions. The easiest to use is that \( V(L) = 0 \):

\[ V(L) = \frac{2w_0}{L} \left( \frac{1}{2} L^2 - \left( L - \frac{L}{2} \right)^2 \right) + C_1 = \frac{2w_0}{L} \left( \frac{L^2}{2} - \frac{L^2}{4} \right) + C_1 = \frac{Lw_0}{2} + C_1 = 0 \quad \implies \quad C_1 = -\frac{Lw_0}{2} \]

so we have

\[ V(x) = \frac{2w_0}{L} \left( \frac{1}{2} x^2 u(x) - \left( x - \frac{L}{2} \right)^2 u \left( x - \frac{L}{2} \right) \right) - \frac{Lw_0}{2} \]
Now we obtain the moment by integrating $V(x)$:

$$M(x) = \int V(x) \, dx$$

$$= \frac{2w_0}{L} x^2 u(x) - \frac{1}{3} x \left( x - \frac{L}{2} \right)^3 u \left( x - \frac{L}{2} \right) - \frac{Lw_0 x}{2} + C_2$$

We find $C_2$ using $M(L) = 0$:

$$M(L) = \frac{2w_0}{L} \left( \frac{1}{6} L^3 - \frac{1}{3} \left( \frac{L}{2} \right)^3 \right) - \frac{L^2 w_0}{2} + C_2$$

$$= \frac{2w_0}{L} \left( \frac{L^3}{12} \right) - \frac{L^2 w_0}{2} + C_2$$

$$= \frac{w_0 L^2}{6} - \frac{L^2 w_0}{2} + C_2$$

$$= - \frac{L^2 w_0}{3} + C_2 = 0 \implies C_2 = \frac{L^2 w_0}{3}$$

so we have

$$M(x) = \frac{2w_0}{L} \left( \frac{1}{3} x^3 u(x) - \frac{1}{3} \left( x - \frac{L}{2} \right)^3 u \left( x - \frac{L}{2} \right) \right) - \frac{Lw_0 x}{2} + \frac{L^2 w_0}{3}$$

We can double-check our answer by evaluating $V(0), M(0)$:

$$V(0) = -\frac{Lw_0}{2}$$

which makes sense: the area under the loading curve is $Lw_0/2$. What about the moment? We have

$$M(0) = \frac{L^2 w_0}{3}$$

Does this make sense? Let’s do a quick integration to find out.

$$M(0) = \int_0^{L/2} \frac{2w_0}{L} x^2 \, dx + \int_{L/2}^L \left( 2w_0 x - \frac{2w_0}{L} x^2 \right) \, dx$$

$$= \frac{2w_0 x^3}{3L} \bigg|_0^{L/2} + w_0 x^2 \bigg|_{L/2}^L - \frac{2w_0 x^3}{3L} \bigg|_{L/2}^L$$

$$= \frac{w_0 L^2}{6} + w_0 L^2 - \frac{w_0 L^2}{4} - \frac{2w_0 L^2}{3} + \frac{w_0 L^2}{12} = \frac{w_0 L^2}{3} \checkmark$$

- Euler-Bernoulli beam theory:

$$\frac{d^2}{dx^2} \left[ E I \frac{d^2 h}{dx^2} \right] = \frac{d^2}{dx^2} \left[ E I \frac{d\theta}{dx} \right] = \frac{d^2}{dx^2} M(x) = \frac{d}{dx} V(x) = w(x) \quad (B.2)$$

- Boundary conditions:

  - Fixed joint: $h(x) = 0, \theta(x) = 0$
  - Free end: $V(x) = 0, M(x) = 0$
  - Pin/pivot end: $h(x) = 0, M(x) = 0$
  - Slider end: $\theta(x) = 0, V(x) = 0$
Example B.4

Continue with the above example. We have an equation for $M(x)$, now integrate to find $\theta(x)$:

$$\theta(x) = \frac{1}{EI} \int M(x) \, dx = \frac{1}{EI} \left[ \frac{2w_0}{L} \left( \frac{1}{12} x^4 u(x) - \frac{1}{12} \left( x - \frac{L}{2} \right)^4 u(x - \frac{L}{2}) \right) - \frac{Lw_0 x^2}{4} + \frac{L^2 w_0 x}{3} + C_3 \right] \quad (B.3)$$

The boundary condition $\theta(0) = 0$ implies $C_3 = 0$. Similarly for the deflection $h(x)$. 

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Principle of Minimum Potential Energy

- Degrees of freedom: the free variables needed to describe the configuration of the system. (Here, we only work with one degree of freedom)
- Potential energy function:
  \[ \Pi(x) = W_{\text{potential}}(x) - W_{\text{applied}}(x) \] (B.4)
- Stationarity condition: the equilibrium condition is found by solving
  \[ \frac{d\Pi}{dx} = 0 \] (B.5)
  for \( x \).
- Potential energy formulae:
  - Linear spring:
    \[ W_{\text{spring}}(x) = \frac{1}{2} k (x - x_0)^2 \] (B.6)
  - Torsion spring:
    \[ W_{\text{torsion}}(\theta) = \frac{1}{2} \kappa \theta^2 \] (B.7)
  - Gravity:
    \[ W_{\text{gravity}}(x) = mgx \] (B.8)
- Energy of applied loads:
  - Applied force
    \[ W_{\text{force}} = fx \] (B.9)
  - Applied moment
    \[ W_{\text{moment}} = M\theta \] (B.10)

Example B.5

A machine made of three pinned beams (all of length \( L \)) is configured in the following way and subjected to two couple moments \( M_1, M_2 \).
The spring is undeformed when $\theta = 0$.

First we construct our energy function. We will have contributions from the spring and from the applied moment $M_2$, but we will not have any contribution from $M_1$ because it does not actually do any work. That is, the member to which it is applied does not rotate. Thus, our energy function is

$$ \Pi(\theta) = \frac{1}{2} k (L \sin \theta)^2 - M_2 \theta \quad (B.11) $$

The stationarity condition is

$$ \frac{d\Pi}{d\theta} = k L^2 (\sin \theta) \cos \theta - M_2 = \frac{1}{2} k L^2 \sin(2\theta) - M_2 = 0 \implies \theta = \frac{1}{2} \sin^{-1} \left( \frac{2M_2}{kL^2} \right) \quad (B.12) $$

where we use a trigonometric identity to simplify. We note that the solution only holds as long as

$$ -1 \leq \frac{2M_2}{kL^2} \leq 1 \quad (B.13) $$

so the maximum moment that can be applied is

$$ M_{\text{max}} = \frac{1}{2} kL^2 \quad (B.14) $$

Now, evaluate stability:

$$ \frac{d^2\Pi}{d\theta^2} = kL^2 \cos(2\theta) \quad (B.15) $$

Substitute the previous value to get

$$ kL^2 \cos \left( \sin^{-1} \left( \frac{2M_2}{kL^2} \right) \right) > 0 \quad (B.16) $$

as our stability criterion. As long as $|M_2| < kL^2/2$ we know that

$$ -\frac{\pi}{2} < \sin^{-1} \left( \frac{2M_2}{kL^2} \right) < \frac{\pi}{2} \quad (B.17) $$

We also know that

$$ \cos \theta > 0 \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2} \quad (B.18) $$
So therefore
\[
kl^2 \cos \left( \sin^{-1} \left( \frac{2M_2}{kl^2} \right) \right) > 0
\] (B.19)
for all \(|M_2| < kl^2/2\), so the solution is stable.

## B.5 Centroid and Moment of Inertia

- **Centroid:** The centroid of a particle is the point at which the weight of the particle can be represented as a single point load. The centroidal coordinates are given by

\[
\bar{x} = \frac{1}{m} \int_{\Omega} x \rho g \, dV \\
\bar{y} = \frac{1}{m} \int_{\Omega} y \rho g \, dV \\
\bar{z} = \frac{1}{m} \int_{\Omega} z \rho g \, dV
\] (B.20)

where
\[
m = \int_{\Omega} \rho g \, dV
\] (B.21)
is the mass of the object and \(\Omega\) is the region of integration.

For 2D objects,
\[
\bar{x} = \frac{1}{m} \int_{\Omega} x \rho g \, t \, dA \\
\bar{y} = \frac{1}{m} \int_{\Omega} y \rho g \, t \, dA
\] (B.22)

where
\[
m = \int_{\Omega} \rho g \, t \, dA
\] (B.23)

For bodies with uniform \(\rho, g, t\), the result is a purely geometric quantity.

- **Centroid of composite objects:** If two bodies (with constant density, thickness) have areas \(A_1, A_2\) and centroidal coordinates \((\bar{x}_1, \bar{y}_1), (\bar{x}_2, \bar{y}_2)\), then the centroidal coordinates for the combined body is

\[
\bar{x} = \frac{\bar{x}_1 A_1 + \bar{x}_2 A_2}{A_1 + A_2} \\
\bar{y} = \frac{\bar{y}_1 A_1 + \bar{y}_2 A_2}{A_1 + A_2}
\] (B.24)

- **2D moment of inertia:** The moments of inertia are

\[
l_{xx} = \int_{\Omega} y^2 \, dA \\
l_{yy} = \int_{\Omega} x^2 \, dA \\
l_{zz} = l_{xx} + l_{yy} \\
l_{xy} = \int_{\Omega} xy \, dA
\] (B.25)

- **Parallel axis theorem:** Let a body have area \(A\), centroidal coordinates \(\bar{x}, \bar{y}\), and moments of inertia \(l_{xx}, l_{yy}\) about the centroidal axes. Then the moments of inertia about (i) a \(y\) axis axes removed a distance \(x_0\) from the centroid and (ii) an \(x\) axis removed a distance \(y_0\) from the centroid is

\[
l'_{xx} = l_{xx} + y_0^2 A \\
l'_{yy} = l_{yy} + x_0^2 A
\] (B.26)

- **3D moment of inertia:** The moments of inertia in 3D are

\[
l_{xx} = \int_{\Omega} (y^2 + z^2) \, dV \\
l_{yy} = \int_{\Omega} (z^2 + x^2) \, dV \\
l_{zz} = \int_{\Omega} (x^2 + y^2) \, dV \\
l_{xy} = \int_{\Omega} xy \, dV \\
l_{yz} = \int_{\Omega} yz \, dV \\
l_{zx} = \int_{\Omega} zx \, dV
\] (B.27)

These make up the moment of inertia tensor:
\[
\mathbf{I} = \begin{bmatrix}
l_{xx} & -l_{xy} & -l_{xz} \\
-l_{yx} & l_{yy} & -l_{yz} \\
-l_{zx} & -l_{zy} & l_{zz}
\end{bmatrix}
\] (B.29)
Example B.6

Consider the following body:

Find

- The centroidal coordinates
- The moments of inertia about the given x,y axes
- The moments of inertia about the centroidal axes

We need to split this problem up into two parts: we will treat the semicircle (body 1) and the rectangle (body 2) separately and then combine the result.

\[
A_1 = \int_0^\pi \int_0^R \rho \, d\rho \, d\theta \\
= \frac{1}{2} \pi R^2
\]

The x component of the centroid is

\[
\bar{x}_1 = \frac{1}{A} \int_0^\pi \int_0^R \rho^2 \cos \theta \, d\rho \, d\theta \\
= \frac{1}{A} \frac{R^3}{3} \int_0^\pi \cos \theta \, d\theta \\
= 0
\]

The y component of the centroid is

\[
\bar{y}_1 = \frac{1}{A} \int_0^\pi \int_0^R \rho^2 \sin \theta \, d\rho \, d\theta \\
= \frac{1}{A} \frac{R^3}{3} \int_0^\pi \sin \theta \, d\theta \\
= \frac{2R^3}{3A} = \frac{4R}{3\pi}
\]

\[
A_2 = \int_0^H \int_{-R}^R \ dx \ dy \\
= 2RH
\]

The x component of the centroid is

\[
\bar{x}_2 = \frac{1}{A} \int_0^0 \int_{-H}^H x \ dx \ dy \\
= \frac{H}{A} \int_{-R}^R x \ dx \\
= \frac{H \ R^2}{A} = 0
\]

The y component of the centroid is

\[
\bar{y}_2 = \frac{1}{A} \int_0^0 \int_{-H}^H y \ dx \ dy \\
= \frac{2R}{A} \int_{-H}^0 y \ dx \\
= -\frac{2R H^2}{A} = -\frac{H}{2}
\]
The centroid of the combined object is

\[
\bar{x} = \frac{x_1 A_1 + x_2 A_2}{A_1 + A_2} = 0 \quad \text{(B.38)}
\]

\[
\bar{y} = \frac{y_1 A_1 + y_2 A_2}{A_1 + A_2} = \frac{(4R/3\pi)(\pi R^2/2) - (H/2)(2RH)}{\pi R^2/2 + 2RH} = \frac{4R^3 - 6RH^2}{3\pi R^2 + 12RH} \quad \text{(B.39)}
\]

Moments of inertia: about the y axis:

\[
l_{yy} = \int_0^\pi \int_0^R \rho^3 \cos^2 \theta \, d\rho \, d\theta \quad l_{y_2y} = \int_{-H}^0 \int_{-R}^R x^2 \, dx \, dy \quad (B.40)
\]

\[
= \frac{R^4}{4} \int_0^\pi \cos^2 \theta \, d\theta \quad = H \int_{-R}^R x^2 \, dx \quad (B.41)
\]

\[
= \frac{1}{8} \pi R^4 \quad = \frac{2HR^3}{3} \quad (B.42)
\]

about the x axis:

\[
l'_{xx} = \int_0^\pi \int_0^R \rho^3 \sin^2 \theta \, d\rho \, d\theta \quad l'_{x_2x} = \int_{-H}^0 \int_{-R}^R y^2 \, dx \, dy \quad (B.43)
\]

\[
= \frac{1}{4} R^4 \int_0^\pi \sin^2 \theta \, d\theta \quad = 2R \int_{-H}^0 y^2 \, dy \quad (B.44)
\]

\[
= \frac{1}{8} \pi R^4 \quad = \frac{2HR^3}{3} \quad (B.45)
\]

To get the combined moments of inertia about the object’s centroid, we use the parallel axis theorem. For the inertia about the y axis we just have

\[
l_y = l'_{yy} + \bar{y}^2 A_1 = \frac{\pi R^4}{8} + \frac{2HR^3}{3} \quad (B.46)
\]

The x case is more complicated. We need to first compute \( l_{1xx}, l_{2xx} \), the moments about the centroidal x axis. Use the parallel axis theorem:

\[
l_{1xx} = l'_{1xx} - \bar{y}_1^2 A_1 = \frac{1}{8} \pi R^4 - \frac{4R^2}{3\pi} \left( \frac{\pi R^2}{2} \right) = \frac{\pi R^4}{8} - \frac{8R^4}{9\pi} \quad (B.47)
\]

\[
l_{2xx} = l'_{2xx} - \bar{y}_2^2 A_2 = \frac{2HR^3}{3} - \left( \frac{H}{2} \right)^2 2RH = \frac{RH^3}{6} \quad (B.48)
\]

Now, we use the parallel axis theorem again to find the moment of inertia about the object's centroid:

\[
l_{xx} = l_{1xx} + (\bar{y}_1 - \bar{y})^2 A + l_{2xx} + (\bar{y}_2 - \bar{y})^2 A \quad (B.49)
\]

We have all of the necessary quantities above, so this is our final solution.

---

**Preview of Dynamics**

We have only begun to introduce the machinery that is needed to model dynamics mathematically. In the next semester of this course, the following topics will be explored:

- **Kinematics**
  - Translation and rotation of a rigid body
- 2D and 3D motion

**Dynamics**
Dynamics is the study of laws of motion using the framework developed in kinematics.

- Dynamics of a particle and a rigid body in linear motion

\[ f = \frac{d}{dt}(mv) \quad \text{(B.50)} \]

- Dynamics of a rigid body in translational motion

\[ M = \frac{d}{dt}(I\omega) \quad \text{(B.51)} \]

**Variational Methods**
Just like we were able to simplify complex problems using the principle of minimum potential energy in statics, we can use energy methods to solve dynamics problems as well. There are two main approaches:

- Lagrangian Mechanics

\[ r(t) = \inf_{r(t)} \int_{t_a}^{t_b} \mathcal{L} \, dt \quad \text{(B.52)} \]

where \( \mathcal{L} = T - V \) is the Lagrangian, \( T \) is the kinetic energy of the system and \( V \) is the potential energy. The solution in one dimension is given by solving the following Euler Lagrange equation:

\[ \frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}} = 0 \quad \text{(B.53)} \]

where \( q \) is position and \( \dot{q} \) is velocity.

- Hamiltonian Mechanics

\[ \dot{q} = \frac{d}{dp} \mathcal{H} \quad \quad \dot{p} = -\frac{d}{dq} \mathcal{H} \]  

\[ \text{(B.54)} \]

where \( \mathcal{H} \) is the Hamiltonian (or total energy of the system), \( p \) is momentum, and \( q \) is position.

**Preview of Strength of Materials**

- Strain and Stress

\[ \varepsilon = \frac{du}{dx} \approx \frac{\Delta x}{x}, \quad \sigma = \frac{df}{da} \quad \text{(B.55)} \]

- Constitutive Relations

  - Hooke’s Law for elongation \( \sigma = E\varepsilon \)
  
  - Thermal stress and strain \( \sigma = E(\varepsilon - \alpha \Delta T) \)

  - Torsion \( M = \frac{1}{2} I_{zz} G \theta \)

- Beam theory

  - Internal stresses

\[ \sigma = \frac{M y}{I_{xx}} \quad \text{(B.56)} \]

- 2D and 3D Stress and Strain
- Stress/strain tensors

\[
\sigma = \begin{bmatrix}
\sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\
\sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\
\sigma_{zx} & \sigma_{zy} & \sigma_{zz}
\end{bmatrix}
\]

\[
\varepsilon = \begin{bmatrix}
\varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\
\varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\
\varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz}
\end{bmatrix}
\]

(B.57)

- Principal stresses and principal strains \( \implies \) eigenvalues of the stress and strain tensors.

- Principal of Minimum Potential Energy

\[
u(x) = \operatorname{arg\, inf}_u \int_\Omega \left( \frac{1}{2} E \varepsilon^2(x) - w(x) u(x) \right) dx
\]

(B.58)